Math 219, Additional Homework Problems

1. The columns of the 3×5 matrix A are all orthogonal to the vector $\vec{v}$. Show that all of the products $A\vec{x}$ must also be orthogonal to $\vec{v}$.

2. (a) Show that for any vector $\vec{a} \in \mathbb{R}^3$, the function $T_{\vec{a}} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_{\vec{a}}(\vec{x}) = \vec{a} \times \vec{x}$ is a linear transformation.
(b) Let $\vec{a} = (1, 2, 3)$. *Without* computing an explicit formula for $T_{\vec{a}}$, find the matrix A that represents $T_{\vec{a}}$ (that is, the matrix A with $T_{\vec{a}}(\vec{x}) = A\vec{x}$).
(c) A matrix representing a cross product in this way must have zeroes as certain elements. Where must these zeroes be?

3. $L : \mathbb{R}^2 \to \mathbb{R}^4$ is linear and we know that $L(1, 0) = (1, 2, 3, 4)$ and $L(0, 1) = (5, 6, 7, 8)$. Find the matrix representing L.

4. We have two linear transformations $S, T : \mathbb{R}^2 \to \mathbb{R}^2$, represented by A and B respectively, and we know that $S(1, 0) = (2, 3)$, $S(0, 1) = (4, 5)$, $T(2, 3) = (6, 7)$, $T(4, 5) = (8, 9)$. Compute BA without computing either A or B.

5. Suppose $z = f(x, y)$ is twice differentiable and x and y are the usual polar coordinate functions of r and θ. Use the chain rule to find an expression for $\frac{\partial^2 f}{\partial \theta^2}$ in terms of only these variables and the partials of z w.r.t. x and y (that is, without any partials of compositions).

6. Suppose $z = f(x, y)$ is twice differentiable, and $x = 3s - 4t$ and $y = 2s + 5t$. Let $w = stxz_x$. Compute $\frac{\partial w}{\partial t}$ in terms of only these variables and the partials of z w.r.t. x and y.

7. The region D is a quadrilateral with vertices at the points $(-2, -2)$, $(3, 1)$, $(1, 3)$, and $(4, 4)$, and we have $f(x, y) = x^2y - y^2x + \sin^5(x - y)$. Compute $\iint_D f(x, y)\, dA$.

8. The area A in the plane $x = 2$ is defined by $y^4 + z^4 = 1$. The solid R is obtained by rotating A around the line defined by $x = 2$ and $y = 3$. We have the function $f(x, y, z) = e^y - e^{6-y}$. Compute $\iiint_R f(x, y, z)\, dV$. *(Hint: A rotated solid is symmetric through every plane containing the axis of rotation.)*

9. Compute (directly from parametrizations) the circulation of the vector field $\vec{F} = (-y, x)$ (a) counterclockwise around the unit circle; (b) clockwise around the unit circle; (c) counterclockwise around the circle of radius 1 centered at $(2, 0)$. (d) Explain why the result of (b) could have been anticipated from that of (a). (e) Explain how the result of (c) is positive even though none of the flow lines go around that circle.

10. The flow on the surface of a river is described by $\vec{F}(x, y) = (4 - \sin(\frac{\pi}{6}(x - y)), 5 + \cos(\frac{\pi}{6}(x + y)))$. Small leaves are falling onto the river; for each leaf, the flow itself causes it to move over the surface, and the circulation on its boundary causes it to spin as it moves. There are currently leaves at $(2, 1)$ and $(3, 2)$; which is moving faster, and which is spinning faster?

11. Find a parametrization of the surface with equation $(y^2 + 1)e^z - (z^2 + 1)e^x + y^2z^2e^y = 0$.

12. Find a parametrization of the surface that results from stretching the unit sphere by a factor of 3 in the y-direction, and then translating it by the vector $(4, 5, 6)$.

13. The parabola P in the xy-plane has equation $y = x^2 + 1$. The surface S is obtained by rotating P around the x-axis. Find a parametrization of S.

14. Let C be the unit circle in the xy-plane, oriented counterclockwise as seen from above. The divergence of the vector field $\vec{F} = (z, x, y)$ is zero, and as a result the flux through every surface with boundary C should be the same. Confirm that this is the case with the upper half of the unit sphere, the lower half of the unit sphere, and the unit disk in the xy-plane (all oriented upward).

15. Compute the flux of the vector field $\vec{F} = (x^2 + 5x - yz, 6 - 2xy, e^y - 5z)$ through the surface S that is the part of the surface $x = y^2 + z^2 - 4$ with $x \leq 0$, oriented in the negative x direction.

16. Compute the flux of the vector field $\vec{F} = (e^z - 2xy, y^2 - e^z, 2xy - y^2)$ through the surface S that is the part of the surface $z = x^4 + e^{y^2}$ with $x + y + z \leq 1$, oriented upward.

17. For a vector field $\vec{F}$, the point $\vec{a}$ is called a "source" if sufficiently small solid balls around $\vec{a}$ have positive boundary flux of $\vec{F}$; and is called a "sink" if sufficiently small solid balls around $\vec{a}$ have negative boundary flux of $\vec{F}$.

Suppose we consider the vector field $\vec{F}(x, y, z) = (x^2y - yz, xy - y^2, xz)$. Is the point $(1, 2, 1)$ a source or a sink for $\vec{F}$? Explain your answer.

18. The flow of a particular kind of pollution in the atmosphere (at a particular time and in a certain vicinity) is given by the vector field $\vec{F}(x, y, z) = (2x - y, x + y - 3z, 2x + 5z)$ (in such a way that the flux of this field through a surface gives the quantity per unit time of the pollution passing through that surface).

Compute the rate of change of the quantity of this pollution in the solid rectangle $[0, 1] \times [2, 3] \times [1, 4]$.

19. A small spherical balloon is held in position at the point $(1, 2, 3)$, but is free to rotate under the influence of the air current described by the vector field $\vec{F}(x, y, z) = (y - z^2, 3x + y + z^2, x^2 - 2y - z)$ giving wind velocity in terms of position.

What is the axis around which the balloon rotates? If the balloon were moved to the point $(0, 0, 0)$, would it rotate faster or slower, and by what factor?

20. On the surface $x^3y - y^3z - z^3x = 43$ at the point $(3, 2, 1)$, compute $\partial y/\partial z$. *(Hint: Keep in mind that a complete argument must first justify the suggestion that y is locally a function of x and z.)*

21. On the surface $x^3y - y^4z - z^5x = 35$ at the point $(3, 2, 1)$, compute $\partial x/\partial y$.