Math 219: Exam 2

3/21/24

This exam contains 9 pages (including this cover page) and 6 questions.

Name: ____

By signing below, you agree to uphold the Duke Community Standard.

Signature: ____

General Rules

- You must show all your work and explain all your reasoning to receive credit. Clarity will be considered when grading.
- No notes, no books, no calculators.
- All answers must be reasonably simplified.

Writing Rules

- Indicate final answers by underlining or boxing them.
- Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.
- Please use black pen.
- Work for a given question can be done only be on the front or back of the page the question is written on. There are three blank pages at the end of the packet.

1. (10 points) Let $a, b > 0$. Find the arc length of the curve parameterized by $(a \cos t, a \sin t, bt)$ for $t \in [0, 2]$.

$$L = \int_0^2 \sqrt{(a(-\sin t))^2 + (a\cos t)^2 + b^2}\, dt$$

$$= \int_0^2 \sqrt{a^2 + b^2}\, dt$$

$$= \boxed{2\sqrt{a^2+b^2}}$$

2. (10 points) For each of the following vector fields, determine whether or not the vector field is a gradient field. If it is a gradient field, find the potential function.

(a) (5 points) $F(x, y, z) = (zx^2, xy, z\sin x)$

Take curl : $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ zx^2 & xy & z\sin x \end{vmatrix}$

$= (0, -z\cos x + x^2, y) \neq 0$

Therefore, not a gradient field.

(b) (5 points) $F(x, y, z) = (y^3e^x + 2xz, 3y^2e^x, x^2)$

Take curl $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^3e^x+2xz & 3y^2e^x & x^2 \end{vmatrix} = (0, 0, 0)$

Therefore, $\vec{F}$ is a gradient field. Find potential Φ

: $\partial_x \Phi = y^3e^x + 2xz$

$\Phi = y^3e^x + x^2z + h(y, z)$

$$\partial_y \Phi = 3y^2 e^x + \partial_y h(y,z) = 3y^2 e^x$$

$$\Rightarrow \quad \partial_y h(y,z) = 0$$

h(z) (no y dependance!)

$$\partial_z \Phi = x^2 + h'(z) = x^2$$

$$\Rightarrow \quad h'(z) = 0$$

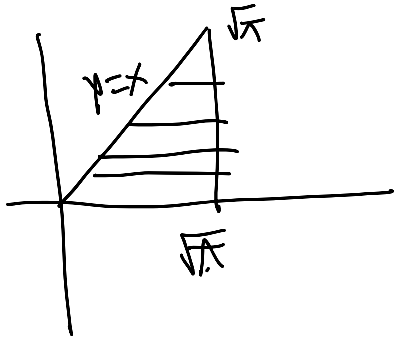
h is a constant

$$\Rightarrow \boxed{\Phi = y^3 e^x + x^2 z + C}$$

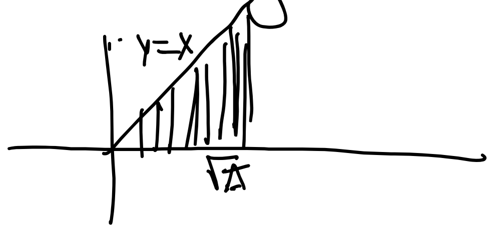
3. (20 points) Evaluate the following integrals.

(a) (10 points)

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2) dx dy.$$



Switch order of integration



$$\int_0^{\sqrt{\pi}} \int_0^{x} \sin(x^2) \, dy \, dx$$

$$= \int_0^{\sqrt{\pi}} \sin(x^2)\, x \, dx = \int_0^{\pi} \sin(x^2) \frac{dx^2}{2} = -\frac{\cos x^2}{2}\Big|_0^{\sqrt{\pi}}$$

$$\boxed{= 1}$$

(b) (10 points) Let D be the the unit disk centered at the origin in the xy plane: $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$,

$$\int_D xy^4 e^{y^2 \cos x} dA.$$

Claim: $f(x,y) \equiv x y^4 e^{y^2 \cos x}$ is odd wrt $x=0$

$R(x,y) = (-x,y)$ $f(-x,y) = -x y^4 e^{y^2 \cos(-x)}$ $= -xy^4 e^{y^2 \cos x}$ $= -f(x,y)$

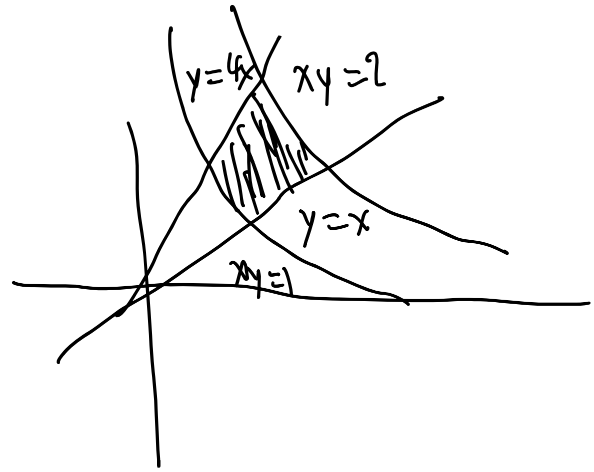
Claim: The disk centered at origin is symmetric wrt $x=0$: since $(-x)^2 + y^2 = x^2 + y^2$

If (x,y) is in disk then so is $(-x,y)$. Therefore integral $\boxed{\text{is } 0}$

4. (20 points) Let D be the region in the first quadrant lying between the hyperbolas $xy = 1$ and $xy = 2$ and the lines $y = x$ and $y = 4x$. Evaluate

$$\int_D x^2 y^3 \, dA.$$

Hint: set $x = u/v, y = uv$.



$x(u,v) = \frac{u}{v}$, $y(u,v) = uv$

Solve for u, v in terms of x, y : $xy = u^2$

$$\frac{y}{x} = v^2$$

Therefore in (u, v) plane, integrate over a rectangle

$1 \leq u^2 \leq 2 \Rightarrow 1 \leq u \leq \sqrt{2}$

$1 \leq v^2 \leq 4 \Rightarrow 1 \leq v \leq 2$

Compute Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$

Now, write integrand x^2y^3 in terms of (u,v)

$$x^2y^3 = \left(\frac{u}{v}\right)^2 (uv)^3 = u^5 v$$

Therefore, putting everything together

$$\int_D x^2y^3 \, dA = \int_1^2 \int_1^{\sqrt{2}} u^5 v \, \frac{2u}{v} \, du \, dv$$

$$= \int_1^2 \int_1^{\sqrt{2}} 2u^6 \, du \, dv$$

$$= 2\int_1^{\sqrt{2}} u^6 \, du$$

$$= \left. \frac{u^7}{7} \right|_1^{\sqrt{2}}$$

$$= \boxed{\frac{2}{7}\left(2^{7/2} - 1\right)}$$

5. (20 points) Let $0 < a < b$. Let D be the region in the first octant between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$. Evaluate

$$\int_D (x + y + z)dV.$$

First, set up integral. Spherical geometry so will use spherical coordinates.

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_a^b (\rho \sin\phi \cos\theta + \rho \sin\phi \sin\theta + \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_a^b \rho^3 \sin^2\phi \cos\theta + \rho^3 \sin^2\phi \sin\theta + \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

Now evaluate each term

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_a^b \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \sin^2\phi \, d\phi \int_0^{\pi/2} \cos\theta \, d\theta \int_a^b \rho^3 \, d\rho$$

$$= \frac{\pi}{4} \times 1 \times \frac{b^4 - a^4}{4} = \frac{\pi(b^4 - a^4)}{16}$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_a^b \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \sin^2\phi \, d\phi \int_0^{\pi/2} \sin\theta \, d\theta \int_a^b \rho^3 \, d\rho$$

$$= \pi/4 \times 1 \times \frac{b^4 - a^4}{4} = \frac{\pi}{16}(b^4 - a^4)$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_a^b \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin 2\phi}{2} \, d\phi \int_0^{\pi/2} d\theta \int_a^b \rho^3 \, d\rho$$

$$= \frac{1}{2} \; \frac{\pi}{2} \; \frac{b^4 - a^4}{4} = \frac{\pi}{16}(b^4 - a^4)$$

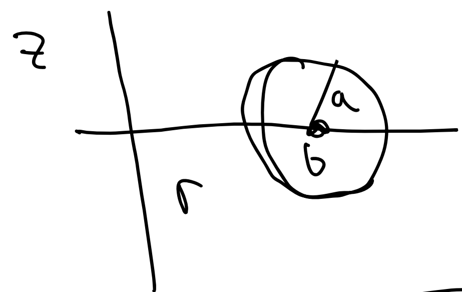
$$\Rightarrow \boxed{\frac{3\pi}{16}(b^4 - a^4)}$$

6. (20 points) We define a region D in $\mathbb{R}^3$ as follows. Let $0 < a < b$. Take the disk in the xz plane of radius a centered at the point $(b, 0, 0)$ and rotate it around the z axis. Let D be defined as the resulting solid of revolution.

(a) (10 points) Set up, but do not evaluate, a multiple integral for the volume of D.

Axis of symmetry ⇒ Probably want to use cylindrical coordinates

Volume of D = integrating 1 over D



To set up integral in rz plane, view z as a function of r using

$$a^2 = z^2 + (r-b)^2 \Rightarrow z = \pm\sqrt{a^2 - (r-b)^2}$$

$$\boxed{\text{Volume of } D = \int_0^{2\pi} \int_{b-a}^{b+a} \int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} dz\, r\, dr\, d\theta}$$

(b) (10 points) Find the volume of D.

$$\text{Volume} = \underbrace{2\pi}_{\theta \text{ integral}} \times \underbrace{2}_{\pm z} \times \int_{b-a}^{b+a} \sqrt{a^2 - (r-b)^2}\, r\, dr$$

$$r - b = u$$

$$= 4\pi \int_{-a}^{a} \sqrt{a^2 - u^2}\,(u+b)\, du$$

$$= 4\pi \left[\int_{-a}^{a} \sqrt{a^2-u^2}\, b\, du + \underbrace{\int_{-a}^{a} \sqrt{a^2-u^2}\, u\, du}_{=0 \text{ by symmetry}} \right]$$

$$= 4\pi b \underbrace{\int_{-a}^{a} \sqrt{a^2 - u^2}\, du}$$

Can do a trig sub or recognize as
the area of a semicircle of radius a

$$= \frac{\pi a^2}{2}$$

$$\Rightarrow \quad \text{Volume } D = 4\pi b \cdot \frac{\pi a^2}{2} = \boxed{2\pi^2 a^2 b}$$

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