Math 219: Exam 1

2/15/24

This exam contains 8 pages (including this cover page) and 5 questions.

Name: ____

By signing below, you agree to uphold the Duke Community Standard.

Signature: ____

General Rules

- You must show all your work and explain all your reasoning to receive credit. Clarity will be considered when grading.
- No notes, no books, no calculators.
- All answers must be reasonably simplified.

Writing Rules

- Indicate final answers by underlining or boxing them.
- Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.
- Use black pen only.
- Work for a given question can be done only be on the front or back of the page the question is written on. There are three blank pages at the end of the packet.

1. (20 points) Let $\vec{a} = (1, 2, 3)$, $\vec{b} = (1, 2, 1)$, $\vec{c} = (0, 1, 1)$. Let $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, $\vec{e}_3 = (0, 0, 1)$.
   (a) Find the projection of $\vec{a}$ onto $\vec{c}$.
   (b) Find a vector orthogonal to $\vec{b}$ and $\vec{c}$.
   (c) Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ as $T\vec{e}_1 = \vec{a}, T\vec{e}_2 = \vec{b}, T\vec{e}_3 = \vec{c}$. Compute $T(0, 2, 1)$.

a) $$\text{Proj}_{\vec{c}}\,\vec{a} = \frac{\vec{a}\cdot\vec{c}}{\|c\|^2}\,\vec{c}$$

$$\vec{a}\cdot\vec{c} = 5, \quad \|c\|^2 = 2$$

$$\frac{5}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} = \boxed{\begin{pmatrix}0\\5/2\\5/2\end{pmatrix}}$$

b) $$\det\begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = i - j - k$$

any scalar multiple of this

$$\boxed{\pm(1, -1, 1)}$$

c) $$T(0,2,1) \overset{\text{linearity}}{=} 0Te_1 + 2Te_2 + Te_3$$
$$= \vec{0} + (2,4,2) + (0,1,1) = \boxed{(2,5,3)}$$

2. (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) = x^3 + y^4 + 2xy$.
   (a) Find the linear approximation of $f(x, y)$ at the point $(x, y) = (1, 1)$.
   (b) Find a parametric representation for the tangent plane of $f(x, y)$ at the point $(x, y) = (1, 1)$.

a) $f(1,1) = 4$

$\partial_x f = 3x^2 + 2y, \quad \partial_x f(1,1) = 5$

$\partial_y f = 4y^3 + 2x, \quad \partial_y f(1,1) = 6$

Liner approximation $L(x,y)$ is

$$\boxed{L(x,y) = 4 + 5(x-1) + 6(y-1)}$$

b) Two tangent vectors at $(1,1)$ are $\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$

(There are others!)

$$\boxed{P(x,y) = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + x\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}}$$

3. (20 points) (a) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ satisfy the conditions $f(\vec{0}) = (1, 2)$ and

$$Df(\vec{0}) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined $g(x, y) = (x + 2y, 3xy)$. Compute $D(g \circ f)(\vec{0})$.

(b) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$D(G \circ F)(\vec{0}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$DF(\vec{0}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $DG(F(\vec{0}))$.

a) $D(g \circ f)(x) = Dg(f(x))\, Df(x)$

$\Rightarrow D(g \circ f)(0) = Dg(f(0))\, Df(0)$

$$Dg(x) = \begin{pmatrix} \partial_x g_1 & \partial_y g_1 \\ \partial_x g_2 & \partial_y g_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3y & 3x \end{pmatrix}$$

$f(0) = (1,2)$

$$Dg((1,2)) = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}$$

$$Dg((1,2)) \cdot Df(0) = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 1 & 2 & 5 \\ 6 & 12 & 21 \end{pmatrix}}$$

b) $D(G\circ F)(\vec{x}) = DG(F(\vec{x}))\,DF(\vec{x})$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = DG(F(0)) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = DG(F(0))$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}}$$

4. (20 points) Let $f(x,y) : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}$ be defined as $f(x,y) = \ln(x^2+y^2)$
   (a) At the point $(1,0)$, what is the direction of the fastest rate of *increase* for f? What is the rate of increase in this direction?
   (b) Compute $(\partial_x^2 + \partial_y^2) f(x,y)$.

a) Fastest direction of increase is the gradient at that point

$$\nabla f(x,y) = \left(\frac{2x}{x^2+y^2} , \frac{2y}{x^2+y^2} \right)$$

$$\nabla f(1,0) = \boxed{(2, 0)}$$

any nonzero multiple of this works

$$\boxed{\|\nabla f\| = 2}$$

b) $$\partial_x^2 f = \partial_x \left(\frac{2x}{x^2+y^2} \right) = \frac{(x^2+y^2)2 - 4x^2}{(x^2+y^2)^2}$$

$$\partial_y^2 f = \partial_y \left(\frac{2y}{x^2+y^2} \right) = \frac{(x^2+y^2)2 - 4y^2}{(x^2+y^2)^2}$$

Simplify:

$$\partial_x^2 f = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}, \quad \partial_y^2 f = \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$$

$$\partial_x^2 f + \partial_y^2 f = \boxed{0}$$

5. (20 points) Define a surface implicitly by the equation $x^2 + 4y^2 + 9z^2 + 3xy = 1$.
   (a) Find a point p so that the tangent plane to the surface at p is parallel to the xy plane.
   (b) Find the distance from the tangent plane you found in part (a) to the xy plane.

a) Normal to the surface given as level set is gradient of equation:

$$\vec{n}(x,y,z) = (2x+3y,\ 2y+3x,\ 18z)$$

Parallel to xy plan means $n(x,y,z)$ is given as a multiple of $(0,0,1)$.

$\Rightarrow$ $2x+3y=0$, $2y+3x=0$ $\Rightarrow$ $(0,0)=(x,y)$ only solution.

Plug into equation $\Rightarrow 9z^2=1 \Rightarrow z=\pm\frac{1}{3}$

So the points are $\boxed{(0,0,\pm\frac{1}{3})}$

b) $\boxed{\text{Distance is } \frac{1}{3}}$ because $(0,0,\pm\frac{1}{3})$ and $(0,0,0)$ lie on a normal vector between two plans.

Scratch. Nothing on this page will be graded.

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