Exam 3: Take Home

By signing below, you agree to uphold the Duke Community Standard.

Signature: ____

General Rules

- You must show all your work and explain all your reasoning to receive credit. Clarity will be considered when grading.
- No outside help except for the professor. This should represent your understanding of the material.
- All answers must be reasonably simplified.

Writing Rules

- Indicate final answers by underlining or boxing them.

Note: For vector fields where I do not specify the domain you can assume that the domain is all of $\mathbb{R}^2$ or $\mathbb{R}^3$ as appropriate.

1. Let C be the closed piecewise smooth curved formed by first following a straight path from $(0,0,0)$ to $(1,0,0)$. Then following $(\cos t, \sin t, 2t/\pi)$ for $t \in [0, \pi/2]$ from $(1,0,0)$ to $(0,1,1)$, and finally following $(0, 1-t, 1/2(1-t)^2 + 1/2(1-t))$ for $t \in [0,1]$ from $(0,1,1)$ back to $(0,0,0)$ in that order. Let $\vec{F} = (2xyz + \sin x, x^2 z, x^2 y)$. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

2. Let C be the closed piecewise smooth curve formed by traveling in straight lines between the points $(1,0,0), (0,1,0), (0,0,1)$ and back to $(1,0,0)$ in that order. Let $\vec{F} = (1,1,1)$. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

3. Let $\vec{F} = (x^2 y + x, y^3 - xy^2)$ and let C_a be the circle of radius a centered at the origin with a counterclockwise orientation and let C_b be the circle of radius b with $0 < b < a$ centered at the origin with a clockwise orientation. Compute

$$\int_{C_a} \vec{F} \cdot d\vec{s} + \int_{C_b} \vec{F} \cdot d\vec{s}.$$

4. Let $\vec{v}$ be a constant vector field and let M be a surface with boundary to which Stokes' theorem applies. Show that

$$2 \int_M \vec{v} \cdot d\vec{S} = \int_{\partial M} \vec{v} \times (x,y,z) \cdot d\vec{s}.$$

5. Let D be the solid described in spherical coordinates as $(\rho, \theta, \phi) \in [0,1] \times [0, 2\pi] \times [0, \pi/3]$. Orient ∂D so that the normal vectors point outward. Compute the flux through ∂D for the following vector fields
   (a) $\vec{F} = (y+z, x+z, x+y)$
   (b) $\vec{F} = (x,y,z)$

6. Let M be the lateral surface of the unit cylinder centered around the z axis between $z = 0$ and $z = 2$ planes. Orient M so that the surface normals point away from z axis (outward facing). Let $\vec{F} = (x^3 z + x^2, -3yx^2 z + z^3, -2zx)$. Compute

$$\int_M \vec{F} \cdot d\vec{S}.$$

7. Define the vector field $F : \mathbb{R}^2 \setminus \{(0,0),(1,0)\} \to \mathbb{R}^2$ as

$$F(x,y) = A\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) + B\left(\frac{-y}{(x-1)^2+y^2}, \frac{x-1}{(x-1)^2+y^2}\right)$$

   (a) Let C be a simple closed curve that does not contain $(0,0)$ or $(1,0)$ given counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   (b) Let C be a simple closed curve that contains $(0,0)$ but does NOT contain $(1,0)$ given counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   (c) Let C be a simple closed curve that contains $(1,0)$ but does NOT contain $(0,0)$ given counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   (d) Let C be a simple closed curve that contains $(1,0)$ and $(0,0)$ given a counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   Hint: Use parts (b) and (c) and try to exploit suitable cancellations between the contours.

Bonus problem

1. Let $\gamma(t) : I \subset \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve whose image lies in the right half plane. Prove that the lateral surface area of the surface of revolution generated by revolving the image of γ around the y axis is equal to $2\pi l(\gamma)\bar{x}$ where $l(\gamma)$ is the length of the image of γ and $\bar{x}$ is defined as

$$\bar{x} = \frac{1}{l(\gamma)} \int_{\gamma(I)} x\, ds$$

   i.e. the average value of x along $\gamma(I)$