Note: For vector fields where I do not specify the domain you can assume that the domain is all of $\mathbb{R}^2$ or $\mathbb{R}^3$ as appropriate.

1. Let C be the closed piecewise smooth curved formed by first following a straight path from $(0,0,0)$ to $(1,0,0)$. Then following $(\cos t, \sin t, 2t/\pi)$ for $t \in [0, \pi/2]$ from $(1,0,0)$ to $(0,1,1)$, and finally following $(0, 1-t, 1/2(1-t)^2 + 1/2(1-t))$ for $t \in [0,1]$ from $(0,1,1)$ back to $(0,0,0)$ in that order. Let $\vec{F} = (2xyz + \sin x, x^2 z, x^2 y)$. Compute
$$\int_C \vec{F} \cdot d\vec{s}.$$

2. Let C be the closed piecewise smooth curve formed by traveling in straight lines between the points $(1,0,0), (0,1,0), (0,0,1)$ and back to $(1,0,0)$ in that order. Let $\vec{F} = (1,1,1)$. Compute
$$\int_C \vec{F} \cdot d\vec{s}.$$

3. Let $\vec{F} = (x^2 y + x, y^3 - xy^2)$ and let C_a be the circle of radius a centered at the origin with a counterclockwise orientation and let C_b be the circle of radius b with $0 < b < a$ centered at the origin with a clockwise orientation. Compute
$$\int_{C_a} \vec{F} \cdot d\vec{s} + \int_{C_b} \vec{F} \cdot d\vec{s}.$$

4. Let $\vec{v}$ be a constant vector field and let M be a surface with boundary to which Stokes' theorem applies. Show that
$$2 \int_M \vec{v} \cdot d\vec{S} = \int_{\partial M} \vec{v} \times (x,y,z) \cdot d\vec{s}.$$

5. Let D be the solid described in spherical coordinates as $(\rho, \theta, \phi) \in [0,1] \times [0, 2\pi] \times [0, \pi/3]$. Orient ∂D so that the normal vectors point outward. Compute the flux through ∂D for the following vector fields
   (a) $\vec{F} = (y+z, x+z, x+y)$
   (b) $\vec{F} = (x,y,z)$

6. Let M be the lateral surface of the unit cylinder centered around the z axis between $z = 0$ and $z = 2$ planes. Orient M so that the surface normals point away from z axis (outward facing). Let $\vec{F} = (x^3 z + x^2, -3yx^2 z + z^3, -2zx)$. Compute
$$\int_M \vec{F} \cdot d\vec{S}.$$

7. Define the vector field $F : \mathbb{R}^2 \setminus \{(0,0),(1,0)\} \to \mathbb{R}^2$ as

$$F(x,y) = A\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) + B\left(\frac{-y}{(x-1)^2+y^2}, \frac{x-1}{(x-1)^2+y^2}\right)$$

   (a) Let C be a simple closed curve that does not contain $(0,0)$ or $(1,0)$ given counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   (b) Let C be a simple closed curve that contains $(0,0)$ but does NOT contain $(1,0)$ given counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   (c) Let C be a simple closed curve that contains $(1,0)$ but does NOT contain $(0,0)$ given counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   (d) Let C be a simple closed curve that contains $(1,0)$ and $(0,0)$ given a counterclockwise orientation. Compute

$$\int_C \vec{F} \cdot d\vec{s}.$$

   Hint: Use parts (b) and (c) and try to exploit suitable cancellations between the contours.

Bonus problem

1. Let $\gamma(t) : I \subset \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve whose image lies in the right half plane. Prove that the lateral surface area of the surface of revolution generated by revolving the image of γ around the y axis is equal to $2\pi l(\gamma)\bar{x}$ where $l(\gamma)$ is the length of the image of γ and $\bar{x}$ is defined as

$$\bar{x} = \frac{1}{l(\gamma(I))} \int_{\gamma(I)} x\, ds$$

   i.e. the average value of x along $\gamma(I)$

#1 First, we compute $\nabla \times \vec{F}$ to check for path independence (since domain is simply connected this is sufficient)

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xyz + \sin x & x^2 z & x^2 y \end{vmatrix}$$

$$= \left(x^2 - x^2, -2xy + 2xy, 2xz - 2xz \right)$$

$$= (0, 0, 0)$$

Therefore $\vec{F}$ is path independent. Since C is a closed curve

$$\boxed{\int_C \vec{F} \cdot d\vec{S} = 0}$$

#2

$\nabla \times \vec{F} = 0$ and C is again a closed curve. Therefore, we have

$$\int_C \vec{F} \cdot d\vec{S} = 0.$$

#3

Let D be the region between C_b and C_a. Green's theorem tells us that

$$\int_{\partial D} \vec{F}\cdot d\vec{S} = \int_D (\nabla\times\vec{F})\cdot\hat{k}\, dA$$

$$\|$$

$$\int_{C_a} \vec{F}\cdot d\vec{S} + \int_{C_b} \vec{F}\cdot d\vec{S}$$ with the orientations given in problem. Therefore, it suffices to compute

$$\int_D (\nabla\times\vec{F})\cdot\hat{k}\, dA$$

$(\nabla \times F)\cdot k = -y^2 - x^2$

To compute the double integral we use polar coordinates

$$\int_D \nabla \times \vec{F} \cdot k \, dA = \int_0^{2\pi}\int_b^a -r^2 \, r \, dr \, d\theta$$

$$= \boxed{2\pi\left(\frac{b^4}{4} - \frac{a^4}{4}\right)}$$

#4

Stoke's theorem states that,

$$\int_{\partial M} \vec{F} \cdot d\vec{s} = \int_{M} \nabla \times \vec{F} \cdot d\vec{S}$$

Let $\vec{F} = \vec{V} \times (x, y, z)$. Thus $\vec{F}$ satisfies the requirements of Stoke's theorem so we have

$$\int_{\partial M} \vec{V} \times (x, y, z) \cdot d\vec{S} = \int_{M} \nabla \times (\vec{V} \times (x, y, z)) \cdot d\vec{S}$$

$$\vec{V} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \vec{V} \times (x, y, z)$$

$$= (bz - cy, \ cx - az, \ ay - bx)$$

$$\nabla \times (\vec{V} \times (x, y, z)) = \begin{vmatrix} i & j & \partial_z \\ \partial_x & \partial_y & \partial_z \\ bz - cy & cx - az & ay - bx \end{vmatrix}$$

$$2(a, b, c) = 2\vec{V} \ \checkmark$$

#5

a) We begin by checking the divergence of $\vec{F}$:

$$\nabla \cdot \vec{F} = \partial_x(y+z) + \partial_y(x+z) + \partial_z(x+y)$$
$$= 0$$

Therefore, by divergence theorem,

$$\int_{\partial D} \vec{F} \cdot d\vec{S} = 0$$

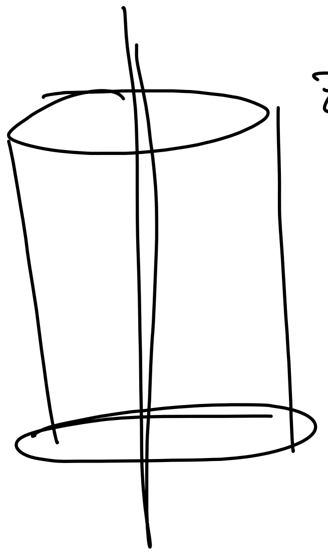
b) In this case $\nabla \cdot F = 3$. Therefore by divergence theorem

$$\int_{\partial D} \vec{F} \cdot d\vec{S} = 3 \int_D dV$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \cdot 2\pi \cdot \tfrac{1}{3} \cdot \tfrac{1}{2} = \boxed{\pi}$$

#6



We first compute $\nabla \cdot \vec{F}$;

$$\partial_x(x^3 z + x^2) + \partial_y(-3yx^2 z + z^3) + \partial_z(-2zx)$$

$$= 3x^2 z + 2x - 3x^2 z - 2x$$

$$= 0$$

Therefore, it seems reasonable to try to apply the divergence theorem. However, because M is not a closed surface, we add "caps" to the top and bottom, denoted as "T" and "B" respectively, so that the union of M, T, B is a closed surface, (where we give T and B outward orientation). Applying the divergence theorem gives

$$\int_M \vec{F} \cdot d\vec{S} + \int_T \vec{F} \cdot d\vec{S} + \int_B \vec{F} \cdot d\vec{S} = 0$$

$$\Rightarrow \int_M \vec{F}\cdot d\vec{S} = -\int_T \vec{F}\cdot d\vec{S} - \int_B \vec{F}\cdot d\vec{S}$$

Therefore, we want to compute

$$\int_T \vec{F}\cdot d\vec{S} \text{ and } \int_B F\cdot d\vec{S}$$

For B we have $d\vec{S} = -\vec{e}_3 dS$:

$$\int_B F\cdot d\vec{S} = \int_{\{z=0\}\cap\{x^2+y^2\leq 1\}} 2zx\, dS = 0$$

For T we have $d\vec{S} = e_3\, dS$

$$\int_T F\cdot d\vec{S} = \int_{\{z=2\}\cap\{x^2+y^2\leq 1\}} -2xz\, dS = \int -4x\, dS$$

$$= \int_0^{2\pi}\int_0^1 -4r^2\cos\theta\, dr\, d\theta$$

$$= \boxed{0} \qquad \int_0^{2\pi}\cos\theta\, d\theta = 0$$

Therefore $\boxed{\int_M \vec{F}\cdot d\vec{S} = 0}$

#7 a) We write the vector field

$\vec{F}$ as $\vec{F}_A + \vec{F}_B$ where

$$\vec{F}_A = A\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

$$\vec{F}_B = B\left(\frac{-y}{(x-1)^2+y^2}, \frac{x-1}{(x-1)^2+y^2}\right)$$

Claim! $\nabla \times \vec{F}_A \cdot \hat{k} = 0 \quad (x,y) \neq (0,0)$

$$\nabla \times \vec{F}_B \cdot \hat{k} = 0 \quad (x,y) \neq (1,0)$$

(I'll just do calculation for F_B since F_A has already been done in lecture 21).

$$\nabla \times \vec{F}_B \cdot \hat{k} = \partial_x\left(\frac{x-1}{(x-1)^2+y^2}\right) - \partial_y\left(\frac{-y}{(x-1)^2+y^2}\right)$$

$$= \frac{(x-1)^2+y^2 - 2(x-1)^2}{((x-1)^2+y^2)^2} + \frac{(x-1)^2+y^2 - 2y^2}{((x-1)^2+y^2)^2} = 0 \checkmark$$

$(x,y) \neq (1,0)$

Therefore, for any closed contour C_1 not containing $(0,0)$ $\int_{C_1} \vec{F}_A \cdot d\vec{S} = 0$

and for any closed contour C_2 not containing $(1,0)$

$$\int_{C_2} \vec{F}_B \cdot d\vec{S} = 0$$ by Green's theorem.

If a closed contour $\vec{C}$ contains neither point then

$$\int_C \vec{F} \cdot d\vec{S} = \int_C \vec{F}_A \cdot d\vec{S} + \int_C \vec{F}_B \cdot d\vec{S}$$

$$= \boxed{0 + 0 = 0}$$

b) In order to compute

$\int_C \vec{F} \cdot d\vec{s}$ with C given in the problem,

~~first~~ observe that

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F}_A \cdot d\vec{s} + \underbrace{\int_C \vec{F}_B \cdot d\vec{s}}_{= 0 \text{ by part a)}}$$

$$= \int_C \vec{F}_A \cdot d\vec{s}$$

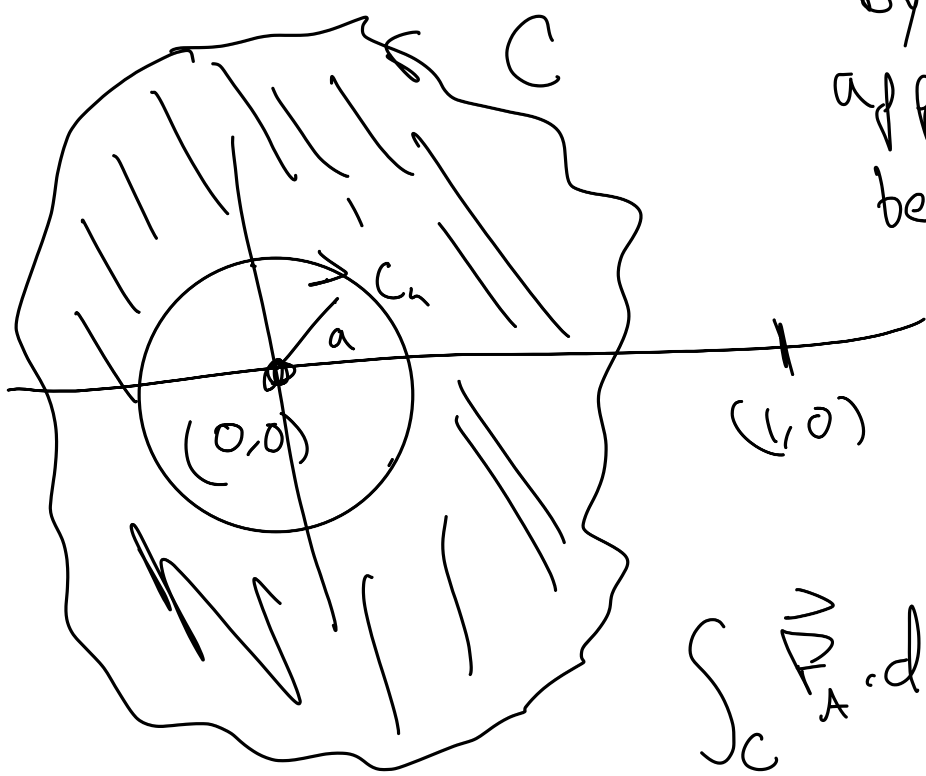
Therefore, it suffices to compute $\int_C \vec{F}_A \cdot d\vec{s}$.
We follow the same procedure outlined in lecture 20
where we first assume C is a circle of
radius $a < 1$ (so doesn't include $(1,0)$)
centered at the origin. For this contour, we
can directly compute that

$$\int_{C_a} \vec{F}_A \cdot d\vec{s} = 2\pi A$$

circle radius a with counter-clockwise parametrization

Now suppose C is a general closed curve containing $(0,0)$ but not containing $(1,0)$.

Since a can be chosen arbitrarily small, we may assume that the circular contour sits inside of C.



By Green's theorem applied to the region between C_a and C we have that

$$\int_C \vec{F}_A \cdot d\vec{s} + \int_{C_a} \vec{F}_A \cdot d\vec{s} = 0$$

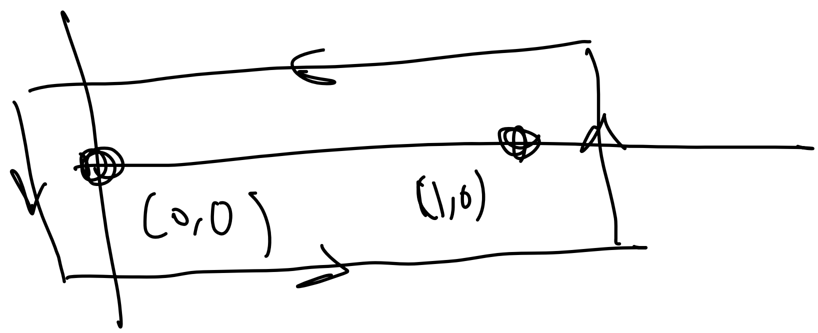
clockwise parameterization

$$\Rightarrow \int_C \vec{F}_A \cdot d\vec{s} = -\int_{C_a} \vec{F}_A \cdot d\vec{s}$$

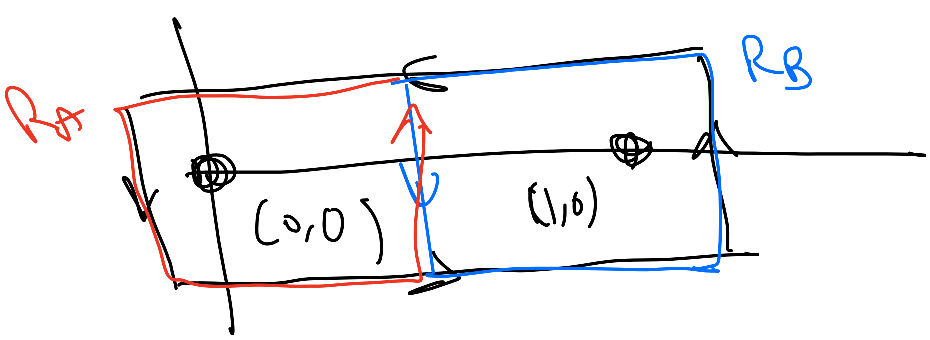
$$\boxed{= 2\pi A}$$

c) Identical to b) except switching roles of $((0,0), \vec{F}_A)$ and $((1,0), \vec{F}_B)$. Answer is $2\pi B$.

D) Following the idea from b), c) we first analyze a simple case. Consider the rectangular contour R drawn below.



We can view R as the sum of two other contours R_A and R_B as follows.



The reason this works is because where R_A and R_B share a side, they have <u>opposite</u> orientations. Therefore

$$\int_R \vec{F} \cdot d\vec{S} = \int_{R_A} \vec{F} \cdot d\vec{S} + \int_{R_B} \vec{F} \cdot d\vec{S}$$

$$= 2\pi A + 2\pi B.$$

For a general contour C, we can use Green's theorem as in part b)

to justify that

$$\int_C \vec{F} \cdot d\vec{S} = \int_R \vec{F} \cdot d\vec{S}$$

Therefore $\boxed{\int_C \vec{F} \cdot d\vec{S} = 2\pi(A+B)}$

Bonus problem

First, note that

$2\pi\, l(\gamma)\, \bar{x}$ can be simplified to

$$2\pi\, l(\gamma) \frac{1}{\cancel{l(\gamma)}} \int_{\gamma(I)} x\, ds = 2\pi \int_{\gamma(I)} x\, ds$$

$$= 2\pi \int_I \gamma_1(t) \| \gamma'(t) \| dt$$

With this goal, we aim to compute the area of the surface of revolution by using the formula

$$A = \int_D \| \vec{N}(s,t) \| \, ds\, dt$$ where $\vec{N} = \partial_s X \times \partial_t X$ and X is a parameterization for the surface of revolution.

We find X by rotating the parameterization $\begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ 0 \end{pmatrix}$ around y axis which is given by the matrix

$$\begin{pmatrix} \cos s & 0 & -\sin s \\ 0 & 1 & 0 \\ \sin s & 0 & \cos s \end{pmatrix} \qquad s \in [0, 2\pi]$$

$$X(s,t) = \begin{pmatrix} \cos s & 0 & -\sin s \\ 0 & 1 & 0 \\ \sin s & 0 & \cos s \end{pmatrix} \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos s\, \gamma_1(t) \\ \gamma_2(t) \\ \sin s\, \gamma_1(t) \end{pmatrix}$$

$$\partial_s X(s,t) = \begin{pmatrix} -\sin s\, \gamma_1(t) \\ 0 \\ \cos s\, \gamma_1(t) \end{pmatrix}$$

$$\partial_t X(s,t) = \begin{pmatrix} \cos s\, \gamma_1'(t) \\ \gamma_2'(t) \\ \sin s\, \gamma_1'(t) \end{pmatrix}$$

$$N(s,t) = \begin{vmatrix} i & j & k \\ -\sin s\,\gamma_1(t) & 0 & \cos s\,\gamma_1(t) \\ \cos s\,\gamma_1'(t) & \gamma_2'(t) & \sin s\,\gamma_1'(t) \end{vmatrix}$$

$$= \left(\gamma_1(t)\gamma_2'(t)\cos s,\ \gamma_1'\gamma_1,\ -\sin s\,\gamma_1\gamma_2'\right)$$

$$\|N(s,t)\| = \sqrt{(\gamma_1\gamma_2')^2 + (\gamma_1\gamma_1')^2}$$

$$= \gamma_1(t)\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} = \gamma_1(t)\,\|\gamma'(t)\|$$

Therefore,

$$\int_D \|N(s,t)\|\,ds\,dt = \int_0^{2\pi}\int_I \gamma_1(t)\,\|\gamma'(t)\|\,ds\,dt$$

$$= 2\pi \int_I \gamma_1(t)\,\|\gamma'(t)\|\,dt.$$

which agrees with the desired result.