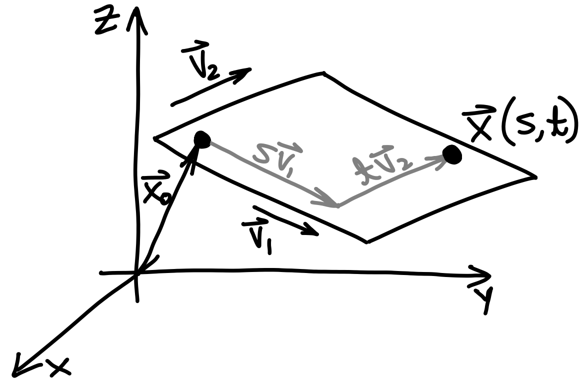
7.1 – Parametrized Surfaces

A parametric <u>surface</u> gives position in $\mathbb{R}^3$ in terms of <u>two</u> parameters (say, s and t).

<u>Ex:</u> $\vec{x}: \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{x}(s,t) = \vec{x}_0 + s\vec{v}_1 + t\vec{v}_2$



This is the plane through $\vec{x}_0$, parallel to $\vec{v}_1$, $\vec{v}_2$.

(See also section 1.5)

Having two parameters, you can't think of them as time. Otherwise like parametric curves.

<u>Graph parametrization</u>

If $z = f(x,y)$, let $x=s$, $y=t$.

<u>Ex:</u> The graph $z = x^2+y^2$ is parametrized by

$$\vec{x}(s,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix}$$

Similarly if $y = f(x,z)$, for example.

<u>Ex:</u> Say $x^3+z^3+ye^{x+z} = 0$.

Note $y = -e^{-(x+z)}(x^3+z^3)$. Letting $x=s$, $z=t$, we parametrize by

$$\vec{x}(s,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -e^{-(s+t)}(s^3+t^3) \\ t \end{pmatrix}$$

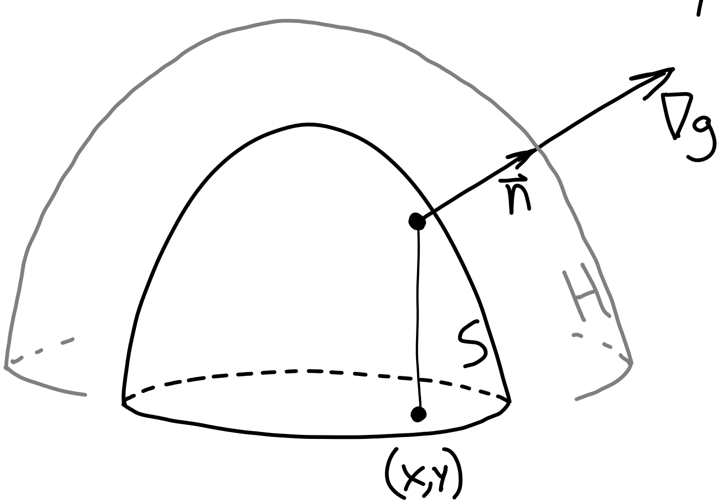
Similarly if you can solve for any coordinate in terms of the others for that coordinate system.

Ex:) Say $\rho = \cos\phi$. Then

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{pmatrix} = \begin{pmatrix} (\cos\phi)\sin\phi\cos\theta \\ (\cos\phi)\sin\phi\sin\theta \\ (\cos\phi)\cos\phi \end{pmatrix}$$

Using vectors

Ex:) $S = \{z = 4 - x^2 - y^2 \geq 0\}$, and at every point there are hairs of length 1 outward $\perp$ to S. Parametrize the surface H formed by the hair ends.



S is parametrized by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ 4 - s^2 - t^2 \end{pmatrix}$$

and is a level set of

$$g = z + x^2 + y^2 - 4.$$

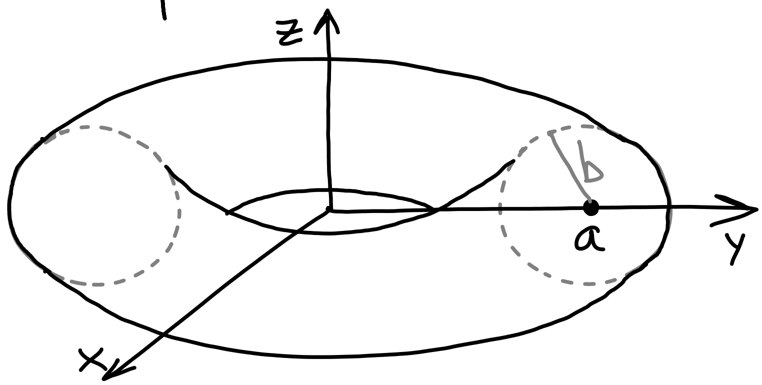
$$\nabla g = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix} = \begin{pmatrix} 2s \\ 2t \\ 1 \end{pmatrix} \quad \text{so} \quad \vec{n} = \begin{pmatrix} 2s \\ 2t \\ 1 \end{pmatrix} \Big/ \sqrt{4s^2 + 4t^2 + 1}$$

Then H is parametrized by

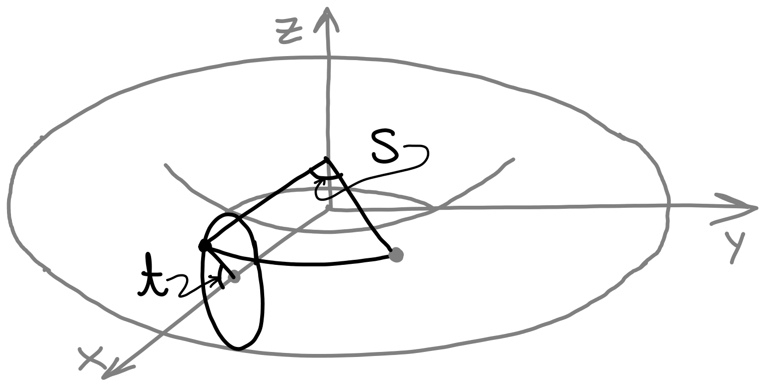
$$\begin{pmatrix} s \\ t \\ 4 - s^2 - t^2 \end{pmatrix} + \begin{pmatrix} 2s \\ 2t \\ 1 \end{pmatrix} \Big/ \sqrt{4s^2 + 4t^2 + 1}$$

Create coordinates

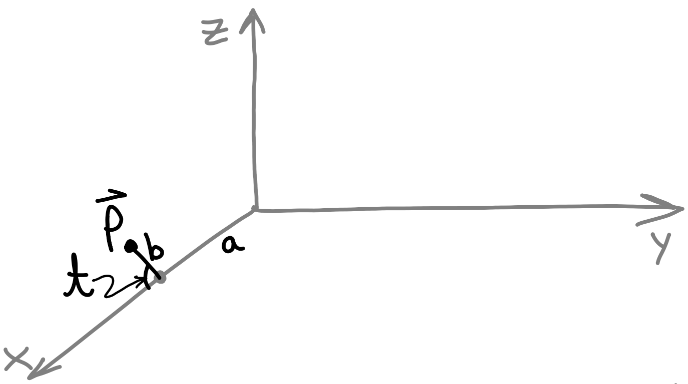
Ex:) How can we parametrize the torus below?



Let s, t be "coordinates" kind of like longitude and latitude:



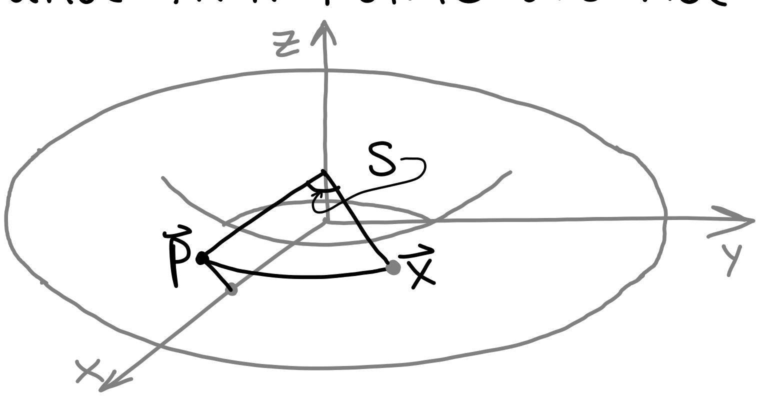
We can then use vectors to compute $\vec{p}$:



$$\vec{p} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} b\cos t \\ 0 \\ b\sin t \end{pmatrix}$$

$$= \begin{pmatrix} a + b\cos t \\ 0 \\ b\sin t \end{pmatrix}$$

and then rotate around the z-axis by angle s with



$$R = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to get

$$\vec{x} = R\vec{p} = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a + b\cos t \\ 0 \\ b\sin t \end{pmatrix}$$

$$= \begin{pmatrix} (\cos s)(a + b\cos t) \\ (\sin s)(a + b\cos t) \\ b\sin t \end{pmatrix}$$

Deform an existing parametrization

Ex: Unit sphere $\rho = 1$ $(x^2+y^2+z^2=1)$ is parametrized by

$$\vec{x} = \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} \qquad \begin{aligned} x &= \sin\phi \cos\theta \\ y &= \sin\phi \sin\theta \\ z &= \cos\phi \end{aligned}$$

so stretching to make the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ we get

$$\vec{x} = \begin{pmatrix} a\sin\phi \cos\theta \\ b\sin\phi \sin\theta \\ c\cos\phi \end{pmatrix} \qquad \begin{aligned} x/a &= \sin\phi \cos\theta \\ y/b &= \sin\phi \sin\theta \\ z/c &= \cos\phi \end{aligned}$$

The book gives many more examples. NB though, most are about interpreting existing parametrizations, not creating them.

On a surface parametrized by $\vec{x}(s,t) = (x,y,z)$:

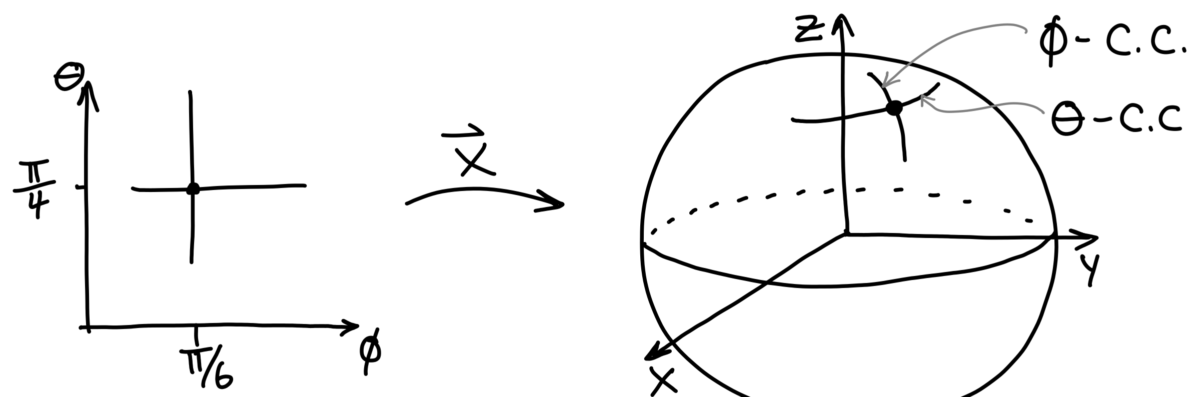
- if you fix t, then $\vec{x}$ is a fn of only s $\Rightarrow$ curve!
- if you fix s, then $\vec{x}$ is a fn of only t $\Rightarrow$ curve!

These are the s- and t- "coordinate curves" of $\vec{x}$.

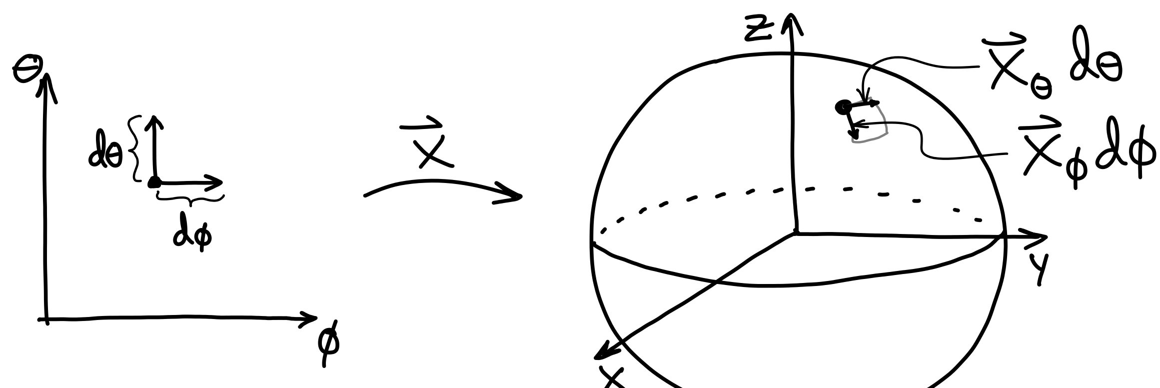
<u>Ex:</u> The unit sphere is parametrized by

$$\vec{x}(\phi,\theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

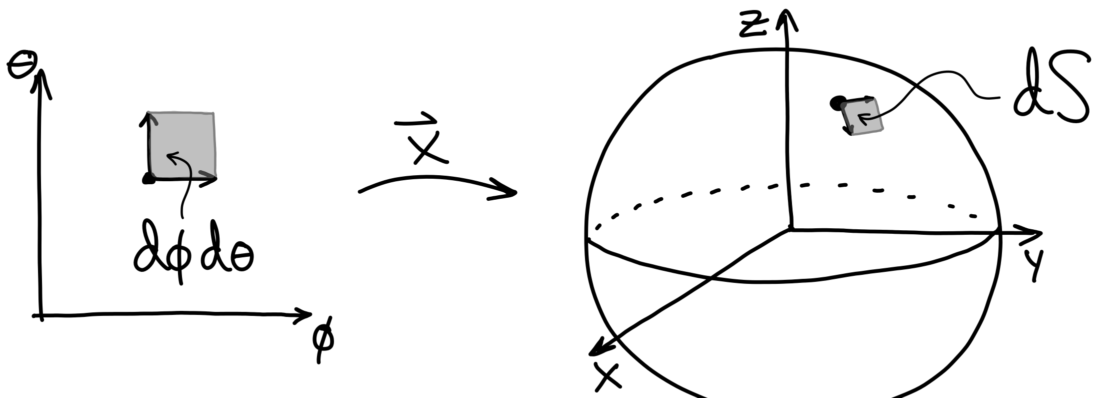
At $\phi = \frac{\pi}{6}$, $\theta = \frac{\pi}{4}$, these are the coordinate curves.



Partial derivatives of $\vec{x}$ are the velocities of these curves. So $\vec{x}_s\,ds$, $\vec{x}_t\,dt$ are the resulting position differentials.



The differential of surface area then is a parallelogram. whose area we can compute.



$$dS = \left\| \vec{X}_\phi \, d\phi \times \vec{X}_\theta \, d\theta \right\| = \left\| \vec{X}_\phi \times \vec{X}_\theta \right\| d\phi \, d\theta$$

<u>Def:</u> For $\vec{X}(s,t)$, the parametrized normal vector is

$$\vec{N} = \vec{X}_s \times \vec{X}_t$$

Alt: "standard"

(Normal ($\perp$) to surface because $\vec{X}_s, \vec{X}_t$ are tangent.)

As above, $\|\vec{N}\|$ is the area stretching factor of $\vec{X}$.

<u>Ex:</u> Compute the area of the unit sphere.

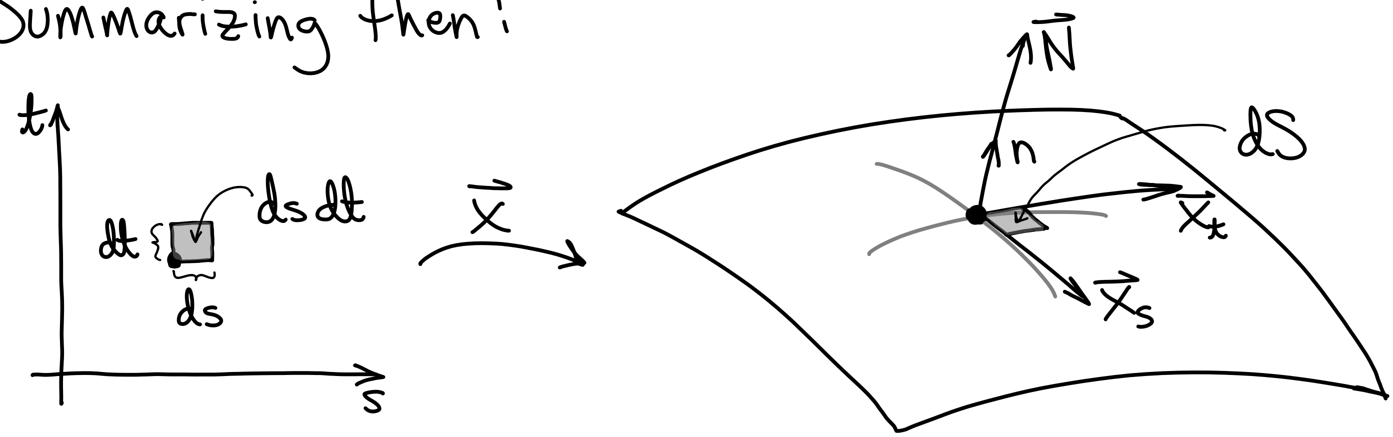
$$\vec{X}(\phi,\theta) = \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} \qquad \vec{X}_\phi = \begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ -\sin\phi \end{pmatrix} \qquad \vec{X}_\theta = \begin{pmatrix} -\sin\phi \sin\theta \\ \sin\phi \cos\theta \\ 0 \end{pmatrix}$$

$$\vec{N} = \vec{X}_\phi \times \vec{X}_\theta = \begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ -\sin\phi \end{pmatrix} \times \begin{pmatrix} -\sin\phi \sin\theta \\ \sin\phi \cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \sin^2\phi \cos\theta \\ \sin^2\phi \sin\theta \\ \sin\phi \cos\phi \end{pmatrix}$$

$$\|\vec{N}\| = \cdots = \sin\phi$$

$$S = \iint dS = \int_0^{2\pi} \int_0^{\pi} \|\vec{N}\| \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta = 4\pi$$

Summarizing then:



$$\vec{N} = \vec{n}\,\|\vec{N}\| = \vec{X}_s \times \vec{X}_t = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} \times \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \partial(y,z)/\partial(s,t) \\ \partial(z,x)/\partial(s,t) \\ \partial(x,y)/\partial(s,t) \end{pmatrix}$$

$$dS = \|\vec{N}\|\, ds\, dt$$

Notice that if we are using a graph parametrization

$$z = f(x,y), \quad \vec{X}(s,t) = (s, t, f(s,t))$$

then

$$\vec{X}_s = \begin{pmatrix} 1 \\ 0 \\ f_s \end{pmatrix} \quad \vec{X}_t = \begin{pmatrix} 0 \\ 1 \\ f_t \end{pmatrix} \quad \vec{N} = \begin{pmatrix} -f_s \\ -f_t \\ 1 \end{pmatrix}$$

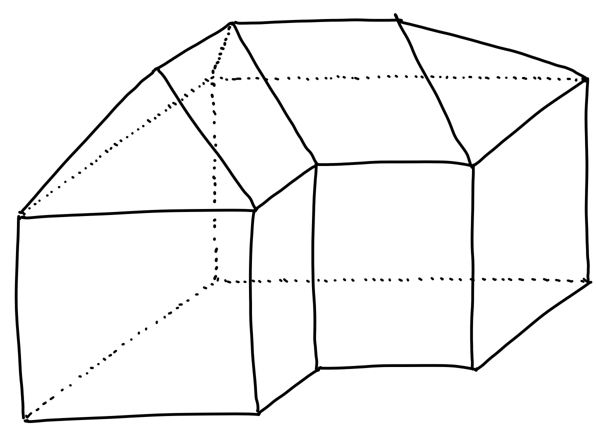
and

$$\|\vec{N}\| = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \|\nabla f\|^2} = \sqrt{1 + (\text{steepness})^2}$$

<u>Ex:</u> A roof is a graph, steepness ~ const.

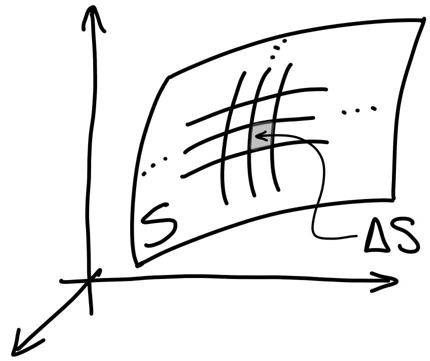
$$S = \iint \sqrt{1 + (\text{steepness})^2}\, dx\, dy$$

$$= \left(\sqrt{1 + (\text{steepness})^2}\right)(\text{footprint area})$$



7.2 - Surface Integrals

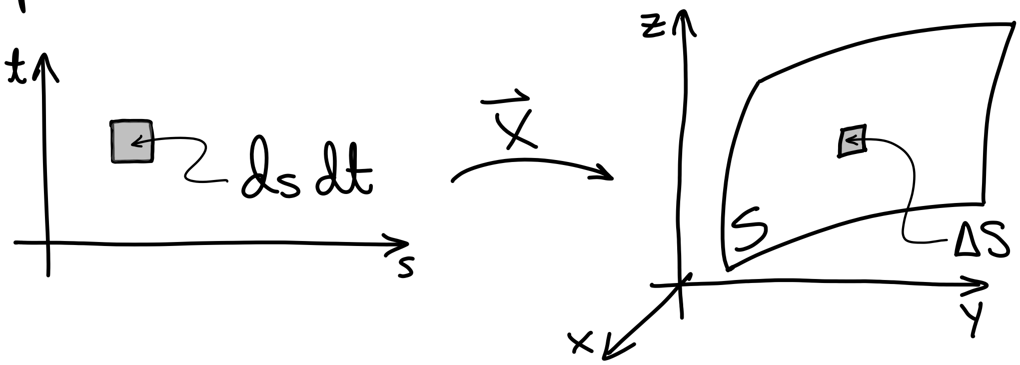
What if we want to do a Riemann sum on a surface (2-d) in space (3-d)?



$$Q = \lim \sum_{i,j} \underbrace{f(\vec{x}_{i,j})}_{\text{density } (Q/\text{area})} \Delta S$$

$$= \iint_S f(\vec{x})\, dS$$ ← scalar surface integral

And we already know how to model this with a parametrization.



$$dS = \|\vec{N}\|\, ds\, dt$$

$$\vec{N} = \vec{X}_s \times \vec{X}_t$$

stretching factor

So we have $$\iint_S f\, dS = \iint_D f(\vec{x}(s,t))\, \|\vec{N}\|\, ds\, dt$$

<u>Handy fact</u>: The above integral is independent of the parametrization. So, use whatever is most convenient!

Ex:) Mass is distributed on the upper unit hemisphere S, with $\delta = z$. Compute the mass, center of mass, and moment of inertia around the z-axis.

$$\vec{x}(\phi,\theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \quad \cdots \Rightarrow \quad \|\vec{N}\| = \sin\phi$$

Then

$$\begin{aligned} m &= \iint_S dm \\ &= \iint_S \delta \, dS \\ &= \int_0^{2\pi}\int_0^{\pi/2} z \|\vec{N}\| \, d\phi \, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} (\cos\phi)(\sin\phi) \, d\phi \, d\theta \end{aligned}$$

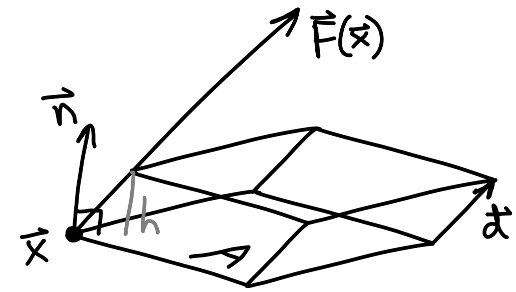
$$\begin{aligned} \bar{x} &= \frac{1}{m}\iint_S x \, dm \\ &= \frac{1}{m}\iint_S x \, \delta \, dS \\ &= \frac{1}{m}\int_0^{2\pi}\int_0^{\pi/2} x \, z \|\vec{N}\| \, d\phi \, d\theta \\ &= \frac{1}{m}\int_0^{2\pi}\int_0^{\pi/2} (\sin\phi\cos\theta)(\cos\phi)(\sin\phi) \, d\phi \, d\theta \end{aligned}$$

$$\begin{aligned} I &= \iint_S r^2 \, dm \\ &= \iint_S r^2 \delta \, dS \\ &= \int_0^{2\pi}\int_0^{\pi/2} (x^2+y^2) \, z \|\vec{N}\| \, d\phi \, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} (\sin^2\phi)(\cos\phi)(\sin\phi) \, d\phi \, d\theta \end{aligned}$$

$$\begin{aligned} \bar{z} &= \frac{1}{m}\iint_S z \, dm \\ &= \frac{1}{m}\iint_S z \, \delta \, dS \\ &= \frac{1}{m}\int_0^{2\pi}\int_0^{\pi/2} z \, z \|\vec{N}\| \, d\phi \, d\theta \\ &= \frac{1}{m}\int_0^{2\pi}\int_0^{\pi/2} (\cos\phi)(\cos\phi)(\sin\phi) \, d\phi \, d\theta \end{aligned}$$

Recall (Ch. 3 notes) that the "flux" (Φ) of $\vec{F}$ through A is

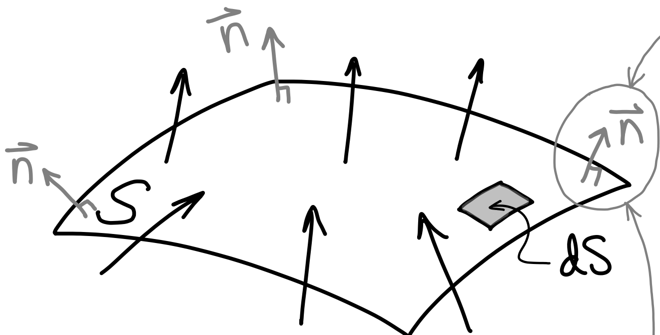
$$\Phi = (\vec{F}\cdot\vec{n})\,A$$



$$\left(\begin{array}{lll} \vec{F} = \vec{v} & \Rightarrow & \Phi = \frac{dV}{dt} \\ \vec{F} = \delta\vec{v} & \Rightarrow & \Phi = \frac{dm}{dt} \end{array} \right)$$

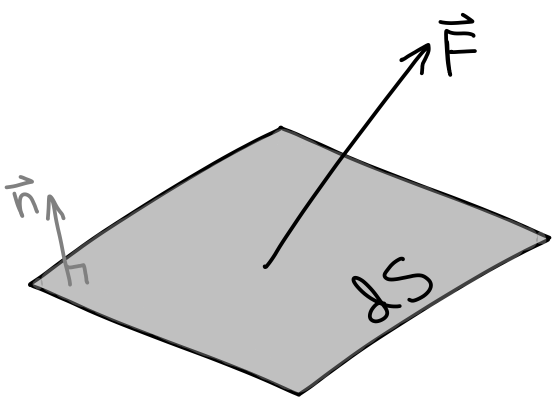
Assumes:
1. $\vec{F}$ is constant
2. A is flat.

But what if $\vec{F}$ varies and the surface is curved?



Let S be an oriented surface.
Chop it into pieces!
1. Each is ~ flat
2. $\vec{F}$ is ~ constant

This $\vec{n}$ is the "orientation unit normal". It defines what counts as the "positive" direction through S.



On each piece,

$$d\Phi = (\vec{F}\cdot\vec{n})\,dS$$

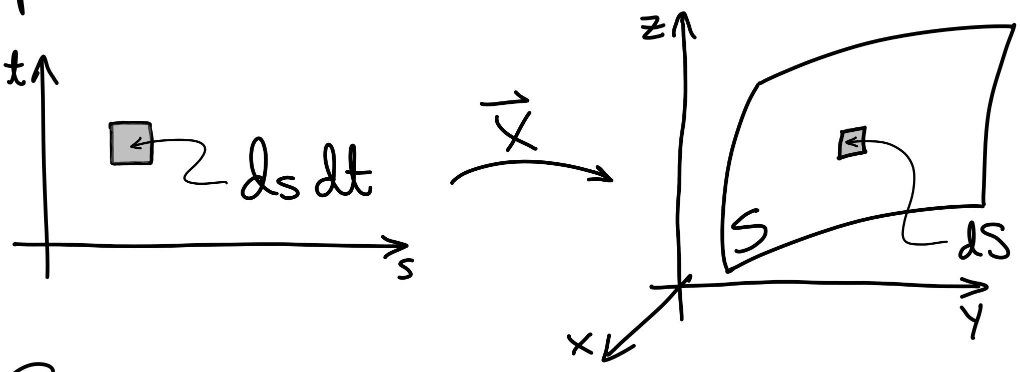
Adding up over all pieces we get

vector surface integral

$$\Phi = \iint_S d\Phi = \iint_S (\vec{F}\cdot\vec{n})\,dS = \iint_S \vec{F}\cdot d\vec{S}$$

($d\vec{S} = \vec{n}\, dS$ encodes both the size (dS) and the orientation ($\vec{n}$) of the differential area!)

And we already know how to model this with a parametrization.



$$dS = \|\vec{N}\|\, ds\, dt$$

$$\vec{N} = \vec{X}_s \times \vec{X}_t$$

stretching factor

So

$$\Phi = \iint_S d\Phi = \iint_S (\vec{F}\cdot\vec{n})\, dS = \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iint_S (\vec{F}\cdot\vec{n})\|\vec{N}\|\, ds\, dt$$

$$= \iint_S \vec{F}\cdot\vec{N}\, ds\, dt$$

*

Ex:) Compute the flux of $\vec{F} = (z^2, x^2, y^2)$ upward through the portion of $z = x^2+y^2$ above the unit square.

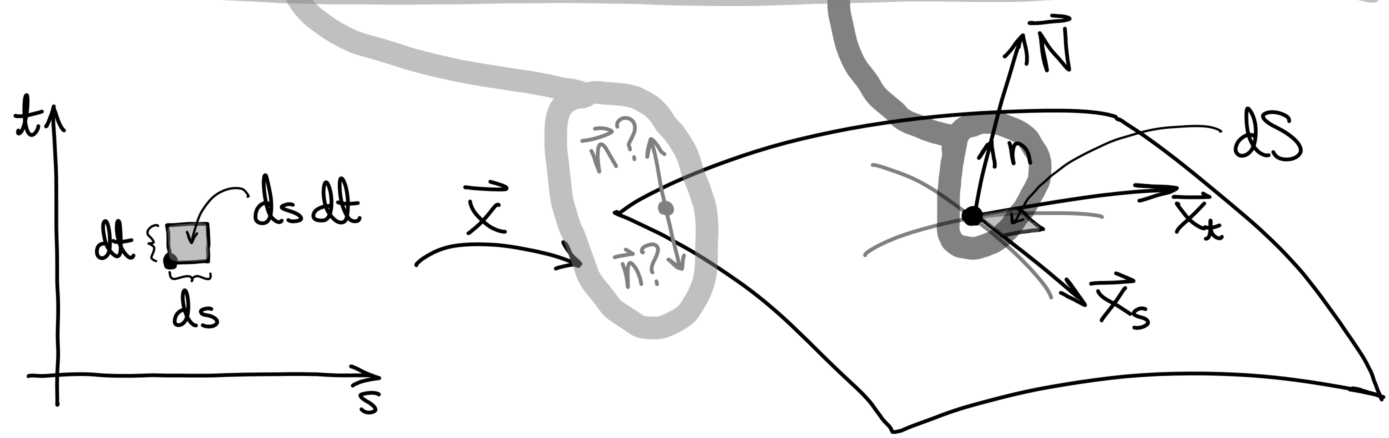
$$\vec{X} = \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix} \quad \vec{X}_s = \begin{pmatrix} 1 \\ 0 \\ 2s \end{pmatrix} \quad \vec{X}_t = \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} \quad \vec{N} = \begin{pmatrix} -2s \\ -2t \\ 1 \end{pmatrix}$$

$$\Phi = \iint \vec{F}\cdot\vec{N}\, ds\, dt = \int_0^1\int_0^1 \begin{pmatrix} (s^2+t^2)^2 \\ s^2 \\ t^2 \end{pmatrix}\cdot\begin{pmatrix} -2s \\ -2t \\ 1 \end{pmatrix} ds\, dt$$

Does the choice of parametrization matter?

Concern:

Is the "orientation unit normal" the same as the "parametrized unit normal"?



We ignored this concern when we wrote:

$$\vdots$$

$$= \iint_S (\vec{F}\cdot\vec{n})\|\vec{N}\|\, ds\,dt$$

this is the orientation unit normal

*

$$= \iint_S \vec{F}\cdot\vec{N}\, ds\,dt$$

$$\vec{N} = \vec{n}\|\vec{N}\|$$

this is the parametrized unit normal

You might have parametrized "upside down"!
Just fix by $\cdot(-1)$!

$$\boxed{d\vec{S} = (\pm?)\vec{N}\,du\,dv}$$

<u>Handy fact:</u> $\iint \vec{F}\cdot\vec{N}\, ds\,dt$ gives the same value as long as $\vec{x}(s,t)$ has the correct orientation.

Ex: Compute the flux of $\vec{F} = (z^2, x^2, y^2)$ downward through the portion of $z = x^2 + y^2$ above the unit square.

$$\vec{X} = \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix} \quad \vec{X}_s = \begin{pmatrix} 1 \\ 0 \\ 2s \end{pmatrix} \quad \vec{X}_t = \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} \quad \vec{N} = \begin{pmatrix} -2s \\ -2t \\ 1 \end{pmatrix}$$

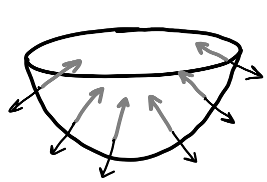
Is our parametrization consistent with the given orientation? No!

We account for this with $\cdot(-1)$:

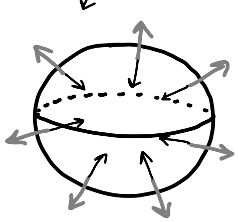
$$\Phi = \iint_S \vec{F} \cdot d\vec{S}$$

$$= -\iint \vec{F} \cdot \vec{N} \, ds\,dt = -\int_0^1 \int_0^1 \begin{pmatrix} (s^2+t^2)^2 \\ s^2 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} -2s \\ -2t \\ 1 \end{pmatrix} ds\,dt$$

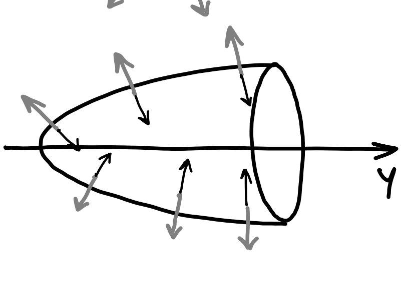
Be careful with the language describing orientations...



upward v. downward ✓
inward v. outward ~✓



upward v. downward ✗
inward v. outward ✓



pos. y-dir v. neg. y-dir. ✓
upward v. downward ✗
inward v. outward ~✓

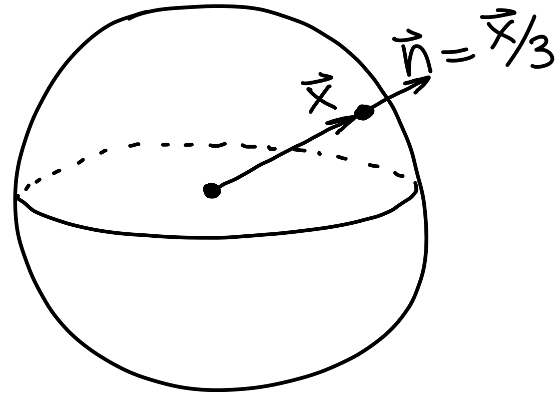
Be on the lookout for lucky coincidences!

Ex: S is the sphere $\rho=3$ oriented outward, and $\vec{F}(\vec{x}) = (yz, -3xz, 2xy)$. Compute $\iint_S \vec{F}\cdot d\vec{S}$.

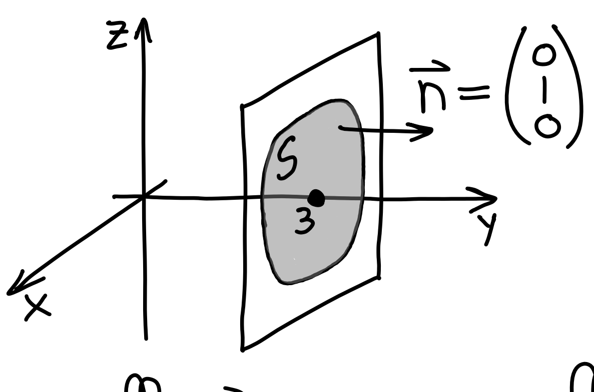
$$= \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= \iint_S \begin{pmatrix} yz \\ -3xz \\ 2xy \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Big/ 3 \, dS$$

$$= 0$$



Ex: S is the part of $y=3$ with $x^4+z^4 \le 1$, oriented in $+y$ direction. $\vec{F}(\vec{x}) = (x^3y - e^z, (x^2+1)(y^2-9), xz^3)$. Compute Φ.



On S, $y=3$,

so $(x^2+1)(y^2-9) = 0$.

$$\text{So } \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \begin{pmatrix} x^3y - e^z \\ 0 \\ xz^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dS = 0$$

Coordinate surface integrals

Recall

$$d\vec{S} = (\pm?)\vec{N}\,ds\,dt = (\pm?)\left(\vec{X}_s \times \vec{X}_t\right)ds\,dt$$

$$= (\pm?)\begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} \times \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} ds\,dt = (\pm?)\begin{pmatrix} \partial(y,z)/\partial(s,t) \\ \partial(z,x)/\partial(s,t) \\ \partial(x,y)/\partial(s,t) \end{pmatrix} ds\,dt$$

It is reasonable to rewrite this as

$$d\vec{S} = (\pm?)\begin{pmatrix} \partial(y,z)/\partial(s,t)\; ds\,dt \\ \partial(z,x)/\partial(s,t)\; ds\,dt \\ \partial(x,y)/\partial(s,t)\; ds\,dt \end{pmatrix} = (\pm?)\begin{pmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{pmatrix}$$

With $\vec{F} = (P, Q, R)$ then, we write

$$\iint_S \vec{F}\cdot d\vec{S} = (\pm?)\iint_S \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \begin{pmatrix} dy\,dz \\ dz\,dx \\ dx\,dy \end{pmatrix}$$

$$= (\pm?)\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

This notation:

- Relates to "differential forms" (Ch. 8).
  Natural & powerful — but beyond this course!
- In common use.

Symmetry theorems

For scalar surface integrals, its just what you would expect:

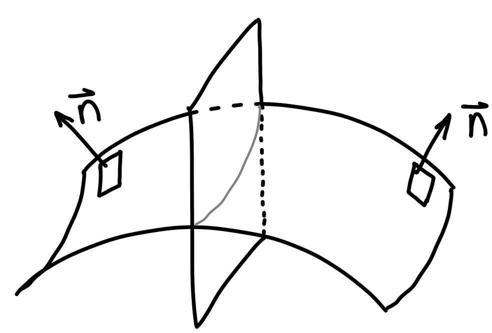
Thm:) If ① f is odd over P
② S is symmetric over P

then $\iint_S f \, dS = 0$

NB – Both conditions must hold!
Must use the same plane for each!

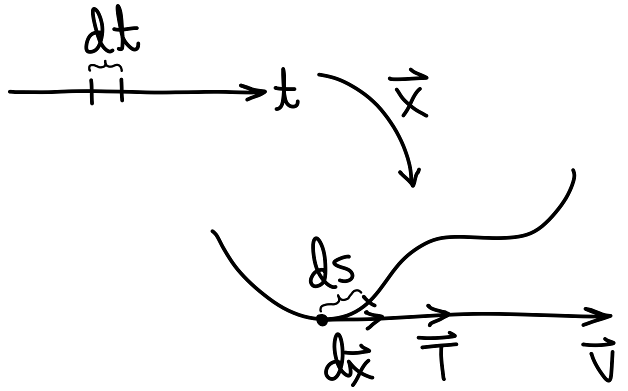
But for vector surface integrals $\iint_S \vec{F} \cdot d\vec{S} = \iint_C \vec{F} \cdot \vec{n} \, dS$, odd symmetry does not work!

(On corresponding pieces of surface, the dS's are equal; but the $\vec{n}$'s are neither equal nor opposite...



Analogies with line integrals

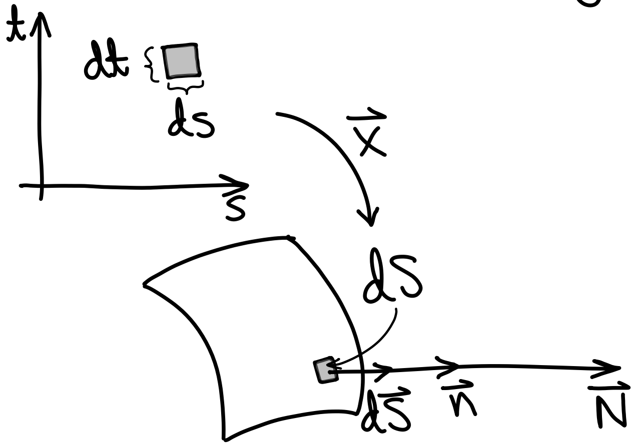
(Vector) line and surface integrals <u>both</u> are examples of differential forms. So there are lots of analogies!



$$\vec{dx} = \vec{T}\,ds = (\pm?)\,\vec{V}\,dt$$

$$\|\vec{dx}\| = ds = \|\vec{V}\|\,dt$$

$$\begin{aligned} \text{Work} &= \int_C dW \\ &= \int_C \vec{F}\cdot\vec{dx} \\ &= \int_C \vec{F}\cdot(\pm?)\,\vec{V}\,dt \\ &= \int_C \vec{F}\cdot\vec{T}\,ds \\ &= \int_C \operatorname{comp}_{\vec{T}}(\vec{F})\,ds \end{aligned}$$



$$\vec{dS} = \vec{n}\,dS = (\pm?)\,\vec{N}\,ds\,dt$$

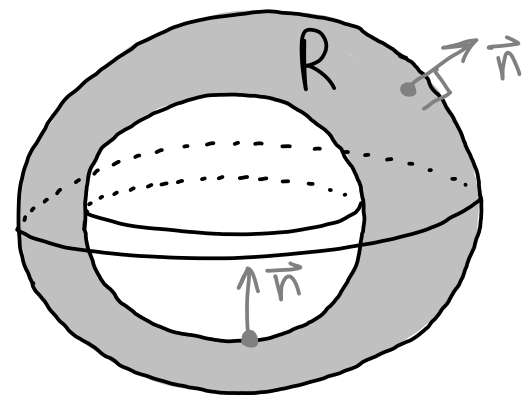
$$\|\vec{dS}\| = dS = \|\vec{N}\|\,ds\,dt$$

$$\begin{aligned} \text{flux} &= \iint_S d\Phi \\ &= \iint_S \vec{F}\cdot\vec{dS} \\ &= \iint_S \vec{F}\cdot(\pm?)\,\vec{N}\,ds\,dt \\ &= \iint_S \vec{F}\cdot\vec{n}\,dS \\ &= \iint_S \operatorname{comp}_{\vec{n}}(\vec{F})\,dS \end{aligned}$$

7.3 - Stokes's and Gauss's Theorems

Gauss's theorem

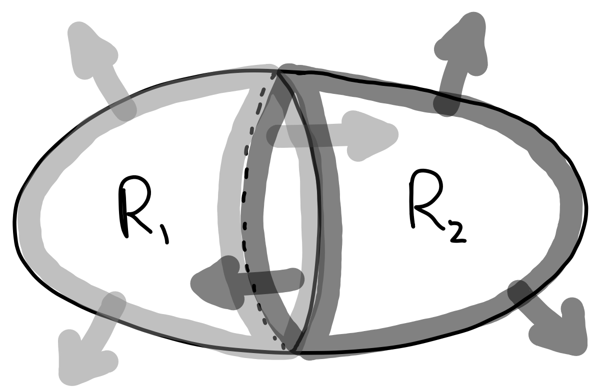
Boundary:
The boundary surface ∂R of a solid region $R \subset \mathbb{R}^3$ is oriented away from the solid.



Accumulation

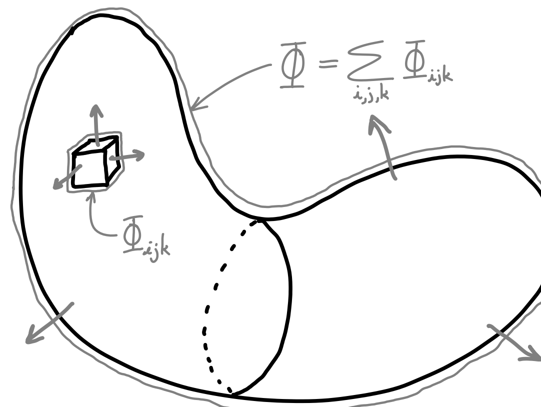
$$\boxed{\text{boundary flux accumulates}}$$

$\iint_{R_1}$, $\iint_{R_2}$ both go over shared face, but with opposite orientations, so they cancel!



(Compare with analogous picture from 6.2, for the 2-d divergence theorem.)

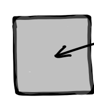
Similarly, it doesn't matter how many pieces, and similarly this accumulation allows for thinking of boundary flux as distributed over the interior, with pieces having location in the interior.

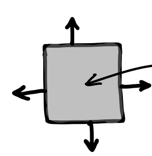


So flux behaves like "stuff".

We <u>measure</u> flux on the boundary — but it actually <u>happens</u> in the interior.
(Compare: F.T.C., Green)

<u>Density</u>

As we define "mass density"...  $dm = \left(\frac{\text{mass}}{\text{volume}}\right) dV$

we also define "flux density":  $d\Phi = \left(\frac{\text{flux}}{\text{volume}}\right) dV$

For $\vec{F} = (P, Q, R)$, flux density $= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
(= "divergence", $\operatorname{div} \vec{F}$, $\nabla \cdot \vec{F}$)

So divergence indicates (per unit volume) how much of the flux measured on the boundary is actually happening at a given point it the interior.

Ex:) $\vec{F} = (x^3, y^3, z^3)$ has a flux Φ through the ball of radius 6. Is more of that flux created at $\vec{a} = (4,0,0)$ or $\vec{b} = (0,3,4)$?

$$\nabla \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\nabla \cdot \vec{F}(\vec{a}) = 48$$

$$\nabla \cdot \vec{F}(\vec{b}) = 75$$

More flux is created at $\vec{b}$

Putting this all together, we get

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \Phi = \iiint d\Phi = \iiint_R \nabla \cdot \vec{F} \, dV$$

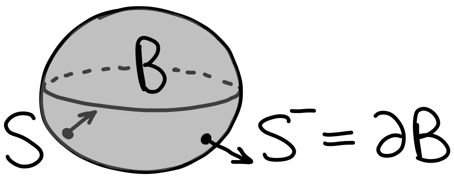
accumulation

density

Gauss's Divergence Theorem

Ex:) S is the inward oriented unit sphere, and $\vec{F}(\vec{x}) = (3x + z^3, \; xy + e^x \sin z, \; 2xy)$. Compute $\iint_S \vec{F} \cdot d\vec{S}$.

Note $S^- = \partial B$, where B is the unit ball. Then

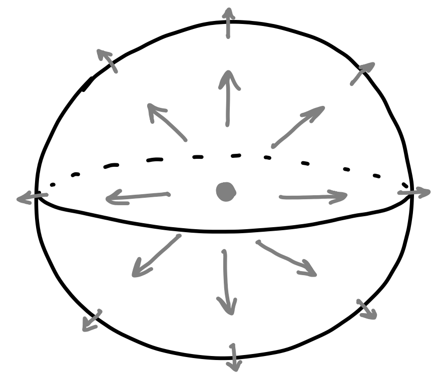


$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_{S^-} \vec{F} \cdot d\vec{S} = -\iint_{\partial B} \vec{F} \cdot d\vec{S} = -\iiint_B \nabla \cdot \vec{F} \, dV$$

$$= -\iiint_B (3 + x) \, dV = -\left(\iiint_B 3 \, dV + \underbrace{\iiint_B x \, dV}_{=0 \text{ by symm.}} \right) = -4\pi$$

Physical examples

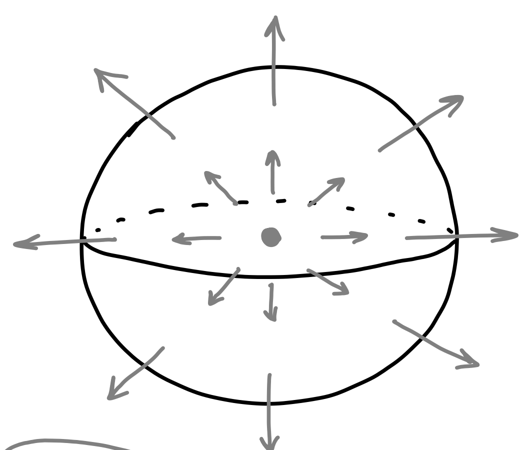
<u>Light bulb</u>: Generates photons, flow like a fluid ($\vec{F}$).
(Note: photon density does not change over time.)



$\nabla\cdot\vec{F} \gg 0$ where the photons are generated. $\nabla\cdot\vec{F} = 0$ everywhere else.

Φ <u>created by</u> "bright spot" where $\nabla\cdot\vec{F} \gg 0$.

<u>Evacuation of a gas</u>: Fluid flowing radially outward, $\vec{F}(\vec{x}) = k\vec{x}$, causing density to be decreasing.
(Note: no fluid being created or destroyed.)



$$\Phi = \iiint_R \nabla\cdot\vec{F}\, dV = \iiint_R (3k)\, dV$$

measured on ∂R indicates mass of fluid leaving R

$\nabla\cdot\vec{F}$ = constant indicates density decreasing same rate everywhere.

(In some cases you have <u>both</u>:

- generation of fluid, and
- changing density of fluid.

Globally: $\Phi + \frac{dm}{dt} = \text{generation rate}$

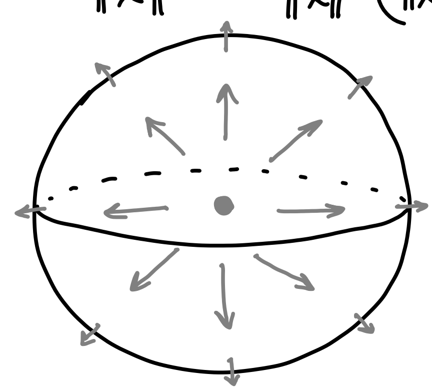
Differentially per unit volume then, we have

Locally: $\nabla \cdot \vec{F} + \frac{d\delta}{dt} = \text{generation rate density}$)

non-physical!

<u>Gravitational fields</u>: For a point mass: $F_g = \frac{k\vec{x}}{\|\vec{x}\|^3} = \frac{k}{\|\vec{x}\|^2}\left(\frac{\vec{x}}{\|\vec{x}\|}\right)$

Inverse square radial — just like a point light source! So, think of gravity as communicated by emitted particles (gravitons) created by mass!



$$(-4\pi G)M = \iiint_R (-4\pi G)\delta \, dV$$

$$\Vert$$

$$\Phi_g = \iiint_R \nabla \cdot \vec{F_g} \, dV$$

All of the gravitons created by mass M flow out of ∂R to make Φ.

Divergence comes from generation density, which comes from mass density

<u>Electric fields</u>: Also inverse square radial! (point source)

charge

charge density

$$\left(\frac{1}{\epsilon_0}\right)Q = \iiint_R \left(\frac{1}{\epsilon_0}\right)\rho \, dV$$

$$\Vert$$

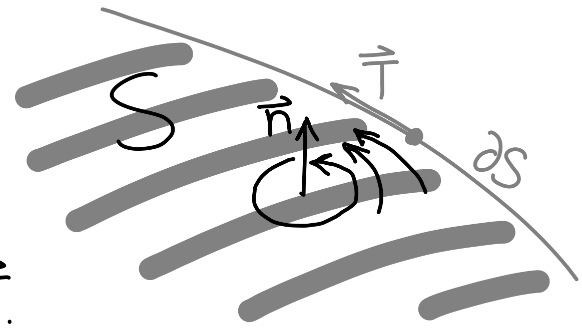
$$\Phi_e = \iiint_R \nabla \cdot \vec{E} \, dV$$

Maxwell's 1st equation (integral form)

Maxwell's 1st equation (differential form)

Stokes's theorem

Boundary: The boundary curve ∂S of an oriented surface $S \subset \mathbb{R}^3$ is oriented by the right hand:

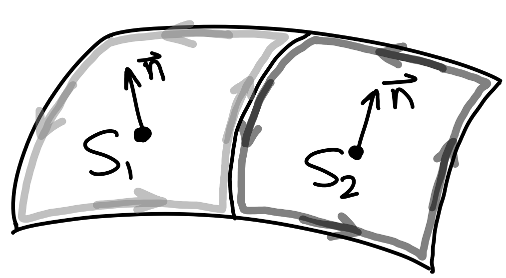


① thumb along surface normal $\vec{n}$.
② fingers define a "swirl" on S.
③ swirl "side swipes" ∂S to give $\vec{T}$.

Accumulation

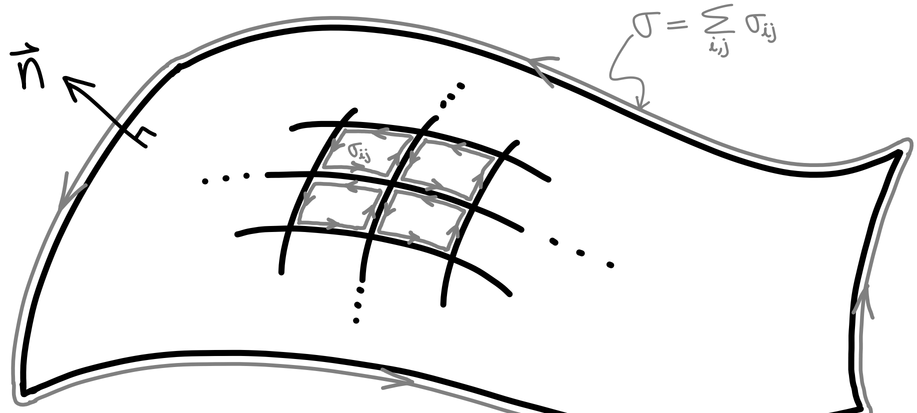
$$\boxed{\text{boundary circulation accumulates}}$$

$\int_{\partial S_1}$, $\int_{\partial S_2}$ both go over shared edge, but with opposite orientations, which then cancel.



(Compare with analogous picture from 6.2, for Green's theorem.)

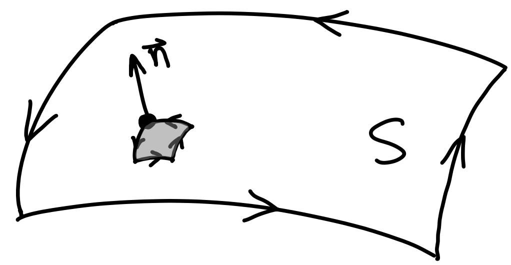
Similarly, it doesn't matter how many pieces.



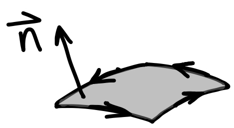
And similarly, this accumulation allows for thinking of this boundary circulation as distributed over the interior, with pieces having location in the interior.

So circulation behaves like "stuff".

We <u>measure</u> circulation on the boundary — but it actually <u>happens</u> in the interior. (Compare: F.T.C.)



<u>Density</u>

Circulation density:  $d\sigma = \left(\frac{\text{circulation}}{\text{area}}\right) dS.$

The formula is: $(\nabla \times \vec{F}) \cdot \vec{n}$. Depends on $\vec{F}$ <u>and</u> S.
(We will make interpretations of this later.)

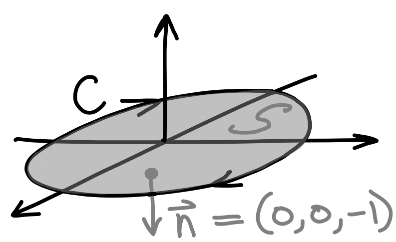
All together then,

$$\int_{\partial S} \vec{F}\cdot d\vec{x} = \sigma \overset{\text{accumulation}}{=} \iint_S d\sigma \overset{\text{density}}{=} \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS$$

Stokes's Curl Theorem $= \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$

<u>Ex:</u> Compute the line integral $\int_C \vec{F}\cdot d\vec{x}$ of $\vec{F} = \left(x^2-y,\ e^y+x,\ \ln(z^2+1)\right)$ on the curve C parametrized by $\vec{x}(t) = (\cos t, -\sin t, 0)$, $t\in[0,2\pi]$.

Note C is a boundary!
(NB orientations!)



$C = \partial S$

Then $$\int_C \vec{F}\cdot d\vec{x} = \int_{\partial S}\vec{F}\cdot d\vec{x} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS$$

$$= \iint_S \begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix} dS = \iint_S (-2)\, dS = (-2)(\text{area}) = -2\pi$$

Oriented density

Recall from FTLI, the (scalar) density depends on both f and C:

$$\Delta = \int_C d\Delta = \int_C (\nabla f \cdot \vec{T})\, ds$$

change density

$$= \int_C \nabla f \cdot d\vec{x}$$

oriented change density

The "f part" is a vector $\nabla f = \|\nabla f\| \vec{u}$.

change density in maximal direction

maximal direction

Similarly for Stokes's theorem, the (scalar) density depends on both $\vec{F}$ and S:

$$\sigma = \iint_S d\sigma = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

circulation density

$$= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

oriented circulation density

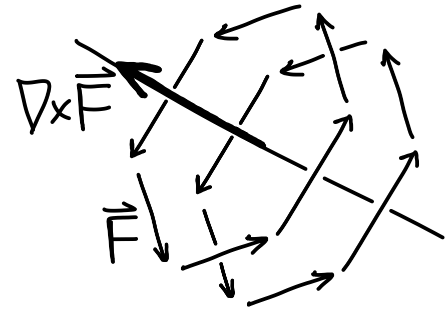
The "$\vec{F}$ part" is a vector $\nabla \times \vec{F} = \|\nabla \times \vec{F}\| \vec{u}$

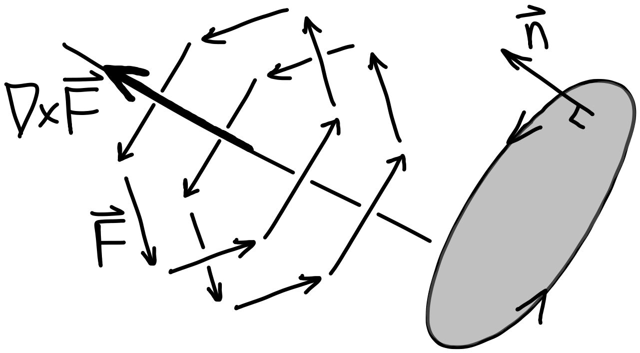
circulation density around rotation axis

direction of rotation axis

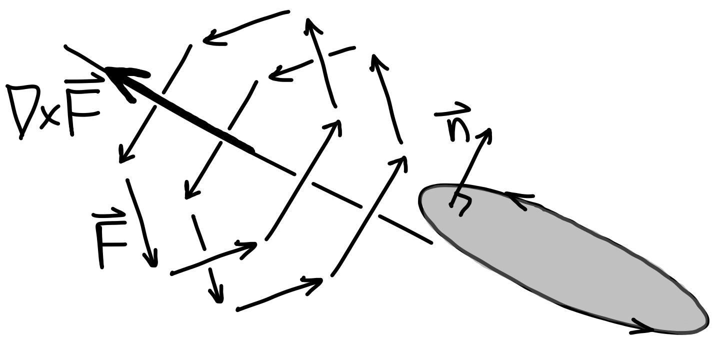
Curl ($\nabla \times \vec{F}$) gives

- the axis of rotation (direction)
- a measure of how much rotation (magnitude).





$\vec{F}$ circulating around boundary corresponds to $\nabla \times \vec{F}$ pointing along $\vec{n}$.



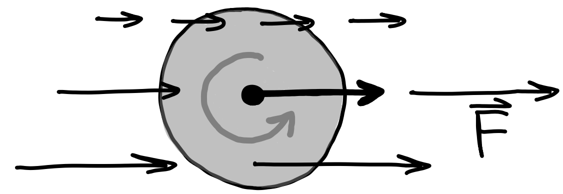
$\vec{F}$ circulating $\perp$ to bdry. corresponds to $\nabla \times \vec{F}$ pointing $\perp$ to $\vec{n}$.

Physical examples

Fluids

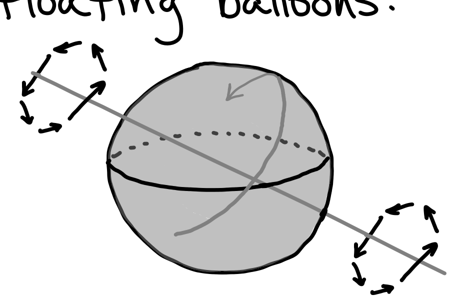
For a fluid in the plane, we can let $\vec{F}(\vec{x}) = \vec{v}$. Then for a floating leaf:

- $\vec{F}$ gives leaf's velocity,
- $\operatorname{grn} \vec{F}$ gives rate of rotation



Similarly in $\mathbb{R}^3$, $\vec{F}(\vec{x}) = \vec{v}$, with floating balloons:

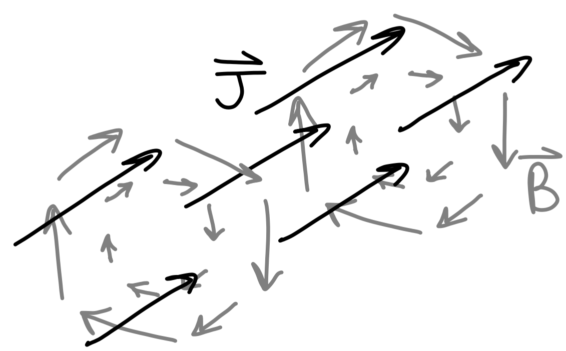
- $\vec{F}$ gives balloon's velocity,
- $\nabla \times \vec{F}$ gives rate and axis of rotation



Magnetic fields

If $\vec{J} = \rho \vec{v}$ is a flow of charged particles (current field), it creates a magnetic field $\vec{B}$.

charge density



$$\begin{aligned} \mu_0 I &= \iint_S (\mu_0 \vec{J}) \cdot d\vec{S} \\ \| \;\; & \\ \sigma_B &= \iint_S (\nabla \times \vec{B}) \cdot d\vec{S} \end{aligned}$$

Maxwell's 4th equation (integral form)

Maxwell's 4th equation (differential form)

A local test

Recall, in $\mathbb{R}^2$:

Thm: For a (C^1) vector field $\vec{F}$ on $\mathbb{R}^2$,

irrotational

$$\left(\vec{F} = \nabla f\right) \iff \left(\operatorname{grn}\vec{F} = 0\right)$$

Pf: $\iff \left(\vec{F} \text{ p.i.}\right) \iff \left(\text{closed } \int_C \vec{F}\cdot d\vec{x} = 0\right) \iff \left(\iint_D \operatorname{grn}\vec{F}\, dA = 0\right) \iff$

Green's Theorem

Analogously, in $\mathbb{R}^3$:

Thm: For a (C^1) vector field $\vec{F}$ on $\mathbb{R}^3$,

$$\left(\vec{F} = \nabla f\right) \iff \left(\nabla \times \vec{F} = \vec{0}\right)$$

Pf: $\iff \left(\vec{F} \text{ p.i.}\right) \iff \left(\text{closed } \int_C \vec{F}\cdot d\vec{x} = 0\right) \iff \left(\iint_D (\nabla\times\vec{F})\cdot d\vec{S} = 0\right) \iff$

Stokes's Theorem

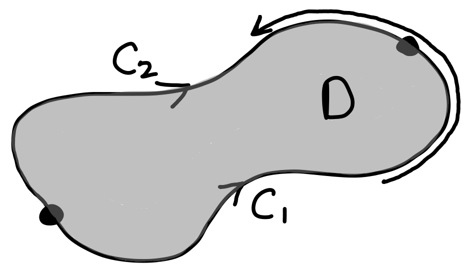
Surface independence

Recall $\vec{F}$ is path independent if

$$(\partial C_1 = \partial C_2) \Rightarrow \left(\int_{C_1} \vec{F}\cdot d\vec{x} = \int_{C_2} \vec{F}\cdot d\vec{x}\right)$$

or/alt

closed $\oint_C \vec{F}\cdot d\vec{x} = 0$



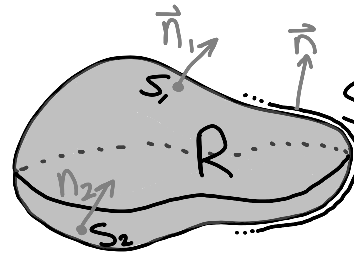
$C = C_1 - C_2 = \partial D$

Similarly, $\vec{F}$ is "surface independent" if

$$(\partial S_1 = \partial S_2) \Rightarrow \left(\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S}\right)$$

or/alt

closed $\oiint_S \vec{F}\cdot d\vec{S} = 0$



$S = S_1 - S_2 = \partial R$

Stokes's theorem shows that every (continuous) curl field $\vec{F} = \nabla\times\vec{G}$ is surface independent!

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \int_{\partial S_1 = \partial S_2} \vec{G}\cdot d\vec{x} = \iint_{S_2} \vec{F}\cdot d\vec{S}$$

Also true (harder to prove): All s.i. fields are curls.

A local test

Thm: For a (C^1) vector field $\vec{F}$ on $\mathbb{R}^3$,

$$(\vec{F} = \nabla\times\vec{G}) \iff (\nabla\cdot\vec{F} = 0)$$

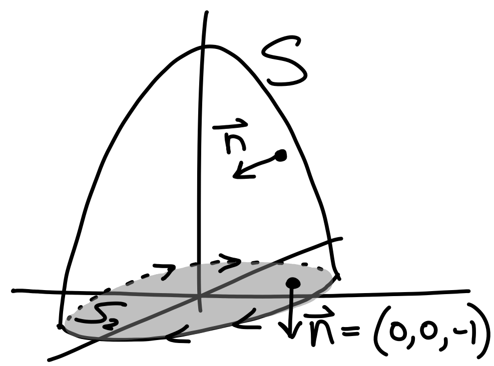
Pf: $\iff (\vec{F}\text{ s.i.}) \iff \left(\text{closed} \oiint_S \vec{F}\cdot d\vec{S} = 0\right) \iff \left(\iiint_D (\nabla\cdot\vec{F})\, dV = 0\right) \iff$

Gauss's Theorem

Ex: $\vec{F} = (y^3, x^3+z, xy)$. S is the part of $z = 36-4x^2-9y^2$ with $z \geq 0$, oriented downward. Compute $\iint_S \vec{F}\cdot d\vec{S}$.

Note $\nabla\cdot\vec{F} = 0+0+0 = 0$

$\Rightarrow \vec{F}$ is s.i.!



S has the same <u>oriented</u> boundary as S_2, shown:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iint_{S_2} (-xy)\, dS$$

$= 0$ by symmetry.

Recall:

Strategic diagram for $\mathbb{R}^2$

$$\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\text{grn}} \text{functions} \quad \Big\} \text{ exact!}$$

$$\text{points} \xleftarrow{\partial} \text{curves} \xleftarrow{\partial} \text{regions} \quad \Big\} \sim\text{exact!}$$

$\underbrace{f(B)-f(A)}_{\Delta} = \int_C \nabla f \cdot d\vec{x}$	$\underbrace{\int_{\partial D} \vec{F}\cdot d\vec{x}}_{\sigma} = \iint_D \text{grn}\vec{F}\, dA$

} same pattern!

Strategic diagram for $\mathbb{R}^3$

$$\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\nabla\times} \text{vector fields} \xrightarrow{\nabla\cdot} \text{functions} \quad \Big\} \text{ exact!}$$

$$\text{points} \xleftarrow{\partial} \text{curves} \xleftarrow{\partial} \text{surfaces} \xleftarrow{\partial} \text{regions} \quad \Big\} \sim\text{exact!}$$

$\underbrace{f(B)-f(A)}_{\Delta} = \int_C \nabla f \cdot d\vec{x}$	$\underbrace{\int_{\partial S} \vec{F}\cdot d\vec{x}}_{\sigma} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$	$\underbrace{\iint_{\partial R} \vec{F}\cdot d\vec{S}}_{\Phi} = \iiint_R \nabla\cdot\vec{F}\, dV$

} same pattern!

Patterns to notice:

- Lifetime theorems (exactness) / Local tests
- Accumulation theorems (FTLI, Stokes, Gauss)
- Path indep. / surf. indep. equiv. properties

Computing line integrals and surface integrals

1. Context from diagram!
2. Compute appropriate operator, boundary, infer consequences.
3. Heads up for useful coincidences, symmetries!

Ex. 1:) $\vec{F} = (y^2+2xz,\ 2xy+z,\ y+x^2)$.
$\vec{x}(t) = (t\cos(\pi t),\ e^{\sin(\pi t)},\ e^{t^2+t})$, $t \in [0,1]$. Compute $\int_C \vec{F}\cdot d\vec{x}$.

Ex. 2: $\vec{F} = \left(x^2 e^x,\ y \sin y,\ e^{z^2}\right)$.

$\vec{x}(t) = \left(t \sin(\pi t),\ e^{\sin(\pi t)},\ e^{t^2 - t}\right)$, $t \in [0,1]$. Compute $\int_C \vec{F} \cdot d\vec{x}$.

Ex. 3: $\vec{F} = (y, z, x)$. $C = \{x+y+z=0\} \cap \{x^2+y^2=1\}$, or. ccw from above.

Compute $\int_C \vec{F} \cdot d\vec{x}$.

Ex. 4: $\vec{F} = (y, z, x)$. $C = \{x - z = 0\} \cap \{x^2 + y^2 = 1\}$, or. ccw from above.
Compute $\int_C \vec{F} \cdot d\vec{x}$.

Ex. 5: $\vec{F} = (xz, -yz, 0)$. $S = \{z = 4 - x^2 - 4y^2,\ z \geqslant 0\}$, or. up.

Compute $\iint_S \vec{F} \cdot d\vec{S}$. (Hint: $\vec{F} = \nabla \times (0, 0, xyz)$.)

Ex. 6: $\vec{F} = (x + y^2 z,\ x^2 z + y,\ xy - z)$. S = unit sphere, or. inward.
Compute $\iint_S \vec{F} \cdot d\vec{S}$.