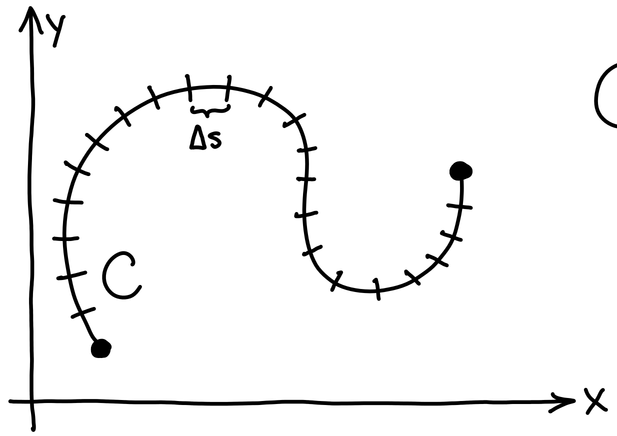
6.1 - Scalar and Vector Line Integrals

Until now, domains have been the same dimension as the space they lived in.

What if our domain is a curve (1-d) in the plane (2-d) or in space (3-d)?

What would a Riemann sum look like?

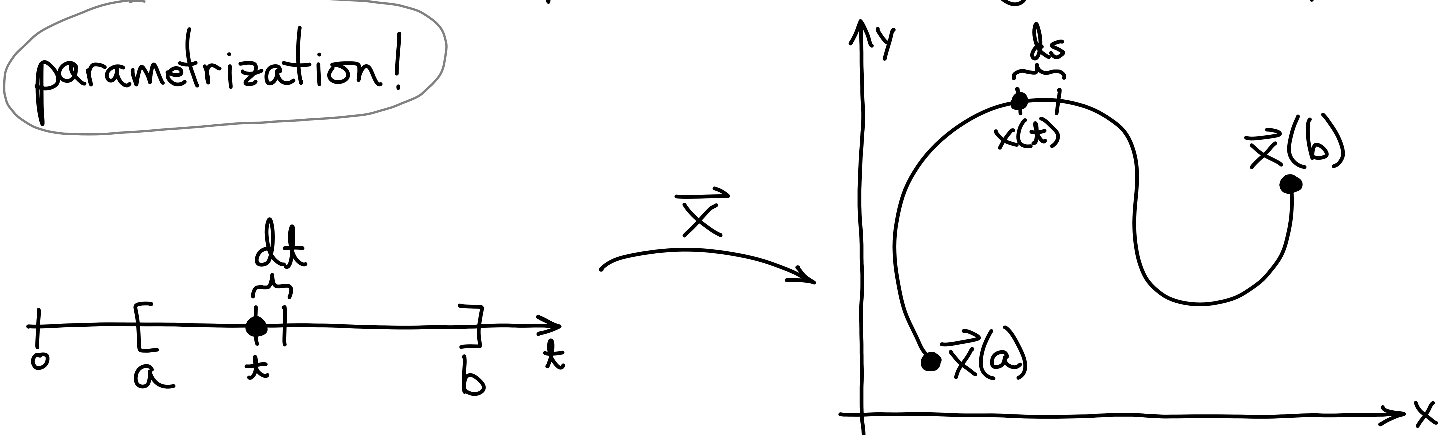


$$Q = \lim \sum_i \underbrace{f(x_i, y_i)}_{\text{density } (Q/\text{length})} \Delta s$$

$$= \int_C f(x,y)\, ds$$ ← scalar line integral

As with change of variables, these are undesirable domains. So let's

1. View as images ← parametrization!
2. Use pull back / stretching factor approach.



We already know that $ds = \|\vec{x}\,'(t)\|\, dt$, so

$$\int_C f(\vec{x})\, ds = \int_a^b f(\vec{x}(t)) \|\vec{x}\,'(t)\|\, dt$$

stretching factor

Handy fact: The above integral is independent of the parametrization. So, use whatever is most convenient!

Ex: An irregular wire follows the curve parametrized by $\vec{x}(t) = (t, t^2)$, $t \in [0,1]$, with $\delta = x + y$.
Compute the mass, center of mass, and m.o.i. around the x-axis.

All of these are done with the same stretching factor!
$\vec{x}' = (1, 2t)$, so s.f. $= \|\vec{x}'\| = \sqrt{1+4t^2}$.

$$\begin{aligned} m &= \int_C dm \\ &= \int_C \delta \, ds \\ &= \int_0^1 (x+y) \, \|\vec{x}'\| \, dt \\ &= \int_0^1 (t+t^2)\sqrt{1+4t^2} \, dt \end{aligned}$$

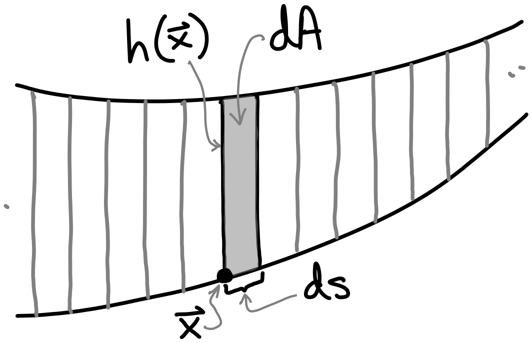
$$\begin{aligned} \bar{x} &= \frac{1}{m}\int_C x \, dm \\ &= \frac{1}{m}\int_C x\,\delta \, ds \\ &= \frac{1}{m}\int_0^1 x\,(x+y) \, \|\vec{x}'\| \, dt \\ &= \frac{1}{m}\int_0^1 (t)(t+t^2)\sqrt{1+4t^2} \, dt \end{aligned}$$

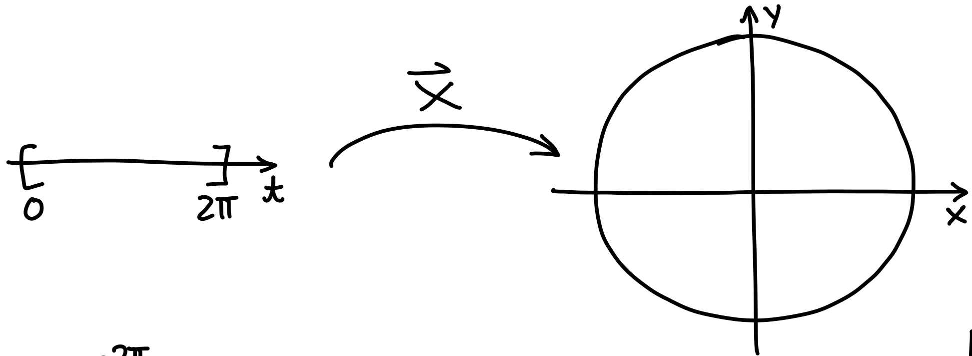
$$\begin{aligned} I &= \int_C r^2 \, dm \\ &= \int_C r^2 \delta \, ds \\ &= \int_0^1 (y)^2 (x+y) \, \|\vec{x}'\| \, dt \\ &= \int_0^1 (t^2)^2 (t+t^2)\sqrt{1+4t^2} \, dt \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m}\int_C y \, dm \\ &= \frac{1}{m}\int_C y\,\delta \, ds \\ &= \frac{1}{m}\int_0^1 y\,(x+y) \, \|\vec{x}'\| \, dt \\ &= \frac{1}{m}\int_0^1 (t^2)(t+t^2)\sqrt{1+4t^2} \, dt \end{aligned}$$

Ex: A fence sits on the unit circle, with $h(x,y) = \frac{x+2}{5}$. What is the area of the fence?

$$A = \int_C dA = \int_C h(\vec{x})\, ds$$





$$\vec{x}(t) = (x, y) = (\cos t, \sin t)$$

$$\vec{x}\,'(t) = (-\sin t, \cos t)$$

$$\|\vec{x}\,'(t)\| = 1$$

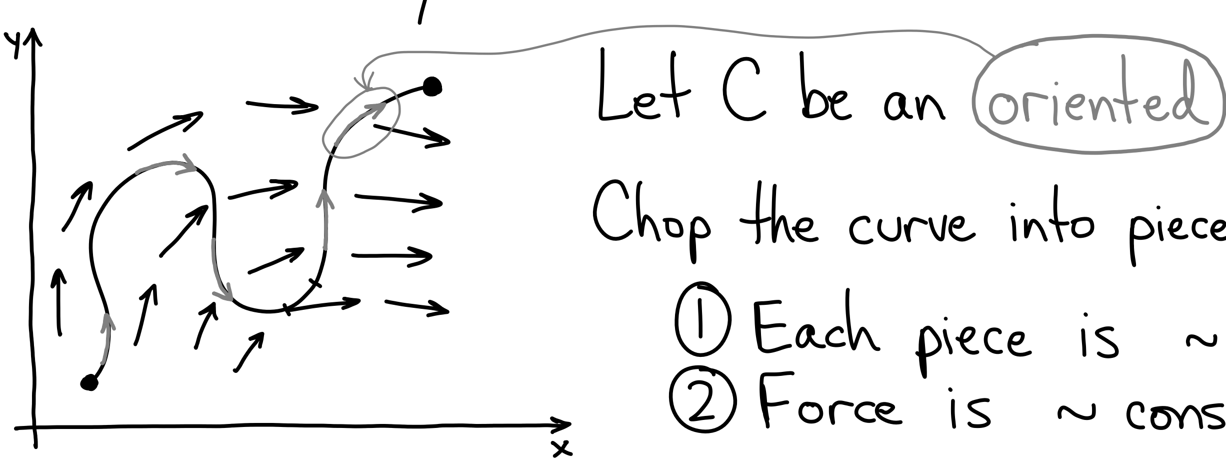
$$= \int_0^{2\pi} \left(\frac{x+2}{5}\right) \|\vec{x}\,'\|\, dt$$

$$= \int_0^{2\pi} \left(\frac{\cos t + 2}{5}\right) (1)\, dt$$

Recall $W = \vec{F} \cdot \vec{d}$. This assumes:

① $\vec{F}$ is constant; ② $\vec{d}$ is over a straight path.

But what if they aren't?

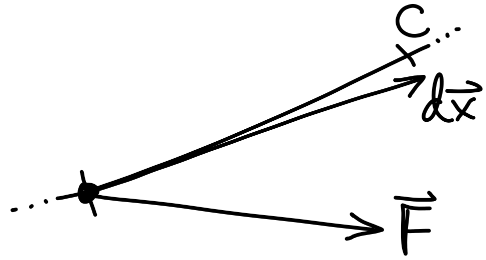


Let C be an oriented curve

Chop the curve into pieces!

① Each piece is ~ straight

② Force is ~ constant



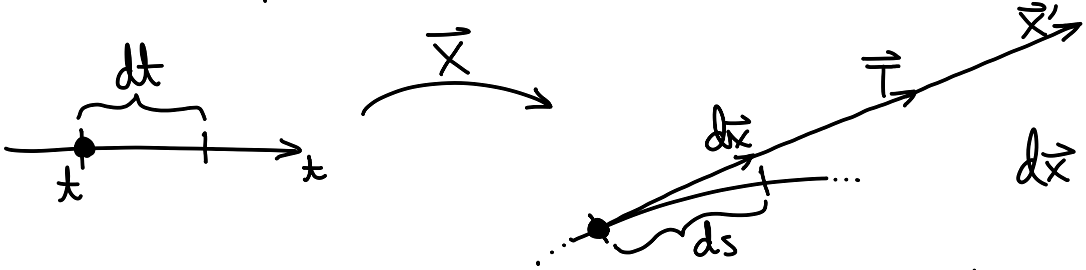
On each piece, $dW = \vec{F} \cdot \vec{dx}$

Adding up over all pieces we get

$$W = \int_C dW = \int_C \vec{F} \cdot \vec{dx}$$

vector line integral

With a parametrization we already know



$$\vec{dx} = \vec{x}' dt = \vec{T} ds$$

so we can rewrite as $\int_C \vec{F} \cdot \vec{dx} = \int_a^b \vec{F} \cdot \vec{x}' \, dt = \int_C \vec{F} \cdot \vec{T} \, ds$

<u>Ex:</u> Wind exerts force on you of $\vec{F}(x,y) = (y, -x)$.
How much work does it take you to walk ccw along the top half of the unit circle?

① <u>You</u> push with force $-\vec{F}$ to oppose the wind. So

$$W = \int_C (-\vec{F}) \cdot \vec{dx}$$

② Parametrize the curve by

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad t \in [0, \pi] \quad \Rightarrow \quad \vec{x}' = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

③ Compute:

$$W = \int_C -\vec{F} \cdot d\vec{x} = \int_0^{\pi} -\vec{F} \cdot \vec{x}'\, dt$$

$$= \int_0^{\pi} -\begin{pmatrix} y \\ -x \end{pmatrix} \cdot \vec{x}'\, dt$$

$$= \int_0^{\pi} -\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt = \int_0^{\pi} 1\, dt = \pi$$

<u>Handy fact:</u> $\int_a^b \vec{F} \cdot \vec{x}'\, dt$ gives the same value as long as $\vec{x}(t)$ has the correct orientation. But...

Key point: $\boxed{\text{The parametrization has to go the same direction as the orientation!}}$

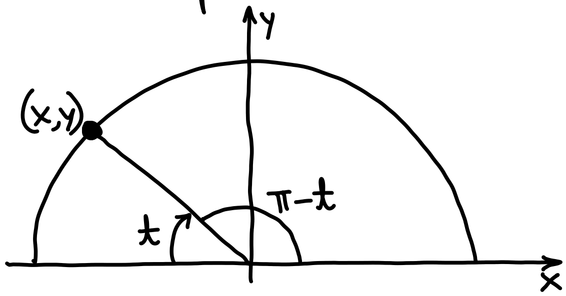
Otherwise your $d\vec{x}$ is backwards, a.k.a. off by $\cdot(-1)$!

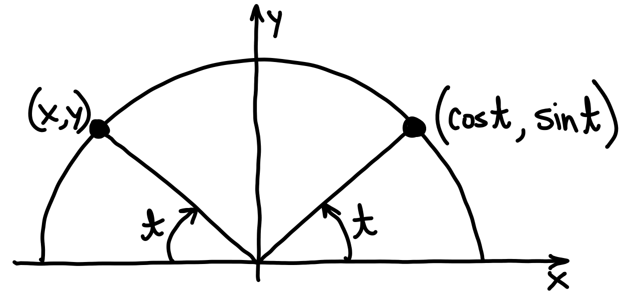
<u>Def:</u> Given an oriented curve C from $\vec{a}$ to $\vec{b}$, the curve C^- moves along the same path with the opposite orientation, from $\vec{b}$ to $\vec{a}$.

<u>Thm:</u> $\int_{C^-} \vec{F} \cdot d\vec{x} = -\int_C \vec{F} \cdot d\vec{x}$

So if you parametrize C the wrong way, your result will be wrong by $\cdot(-1)$.

Ex:) Same as previous, what if walking cw?
How to parametrize?





$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\pi - t) \\ \sin(\pi - t) \end{pmatrix} = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Then $$W = \int_{C_2} (-\vec{F}) \cdot d\vec{x} = \cdots = -\pi$$

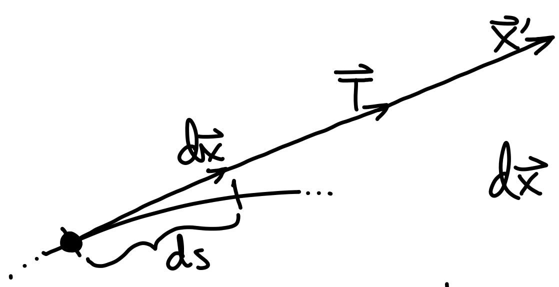
Alt: This is the same curve as previous with the opposite orientation. So

$$\int_{C_2} = \int_{C^-} = -\int_C = -\pi$$

We have seen that if $\vec{F}$ represents forces, then $\int_C \vec{F} \cdot d\vec{x}$ represents work.

What if $\vec{F}$ represents a fluid flow (as velocity)?

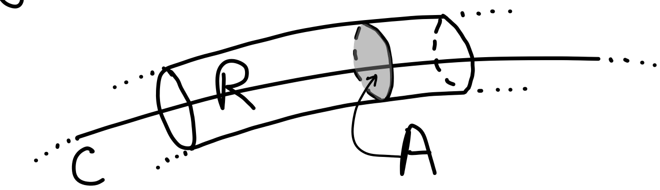
Recall:



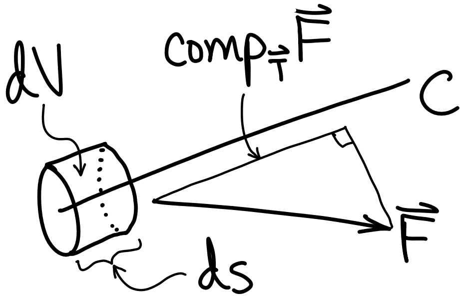
$$d\vec{x} = \vec{x}' dt = \vec{T} ds$$

$$\int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F} \cdot \vec{x}' \, dt = \int_C \vec{F} \cdot \vec{T} \, ds$$

Consider a thin region R around C, with section area A.



What is the total rate Q that the fluid in R is flowing along C?

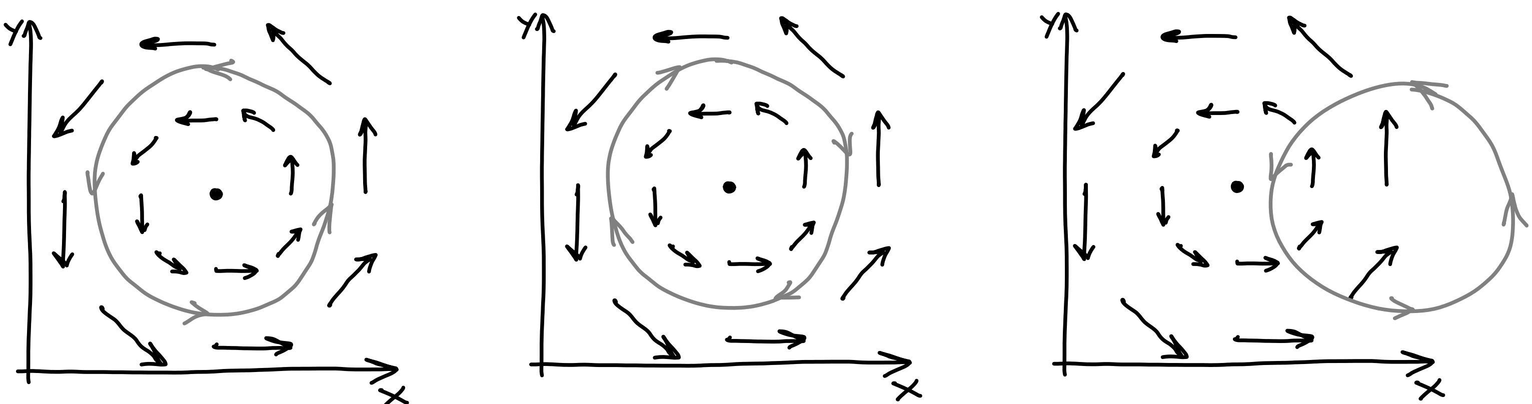


$$dQ = \left(\text{comp}_{\vec{T}}\vec{F}\right) dV$$
$$= \left(\vec{F}\cdot\vec{T}\right)(A\, ds)$$

$$Q = \int_C dQ = A\int_C \vec{F}\cdot\vec{T}\, ds = A\int_C \vec{F}\cdot d\vec{x}$$

So $\int_C \vec{F}\cdot d\vec{x}$ represents the rate (per section area) of fluid flowing along C.

For a closed curve (starts/ends at same point), $\int_C \vec{F}\cdot d\vec{x}$ is called the <u>circulation</u> (σ) of $\vec{F}$ along C.

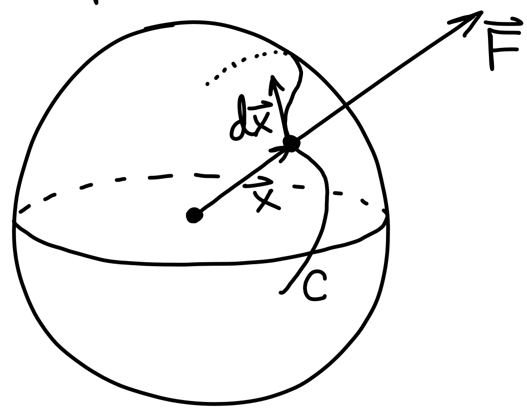


$\sigma > 0$ $\qquad$ $\sigma < 0$ $\qquad$ $\sigma = ?$

For small objects, circulation suggests how the fluid would rotate it.

Look out for fortunate geometry!

Ex: C is parametrized by $\rho=1$, $\phi = t\sin t$, $\theta = t^2 e^t$ for $t\in[0,1]$, and $\vec{F} = x^2 y\,\vec{x}$. Compute $\int_C \vec{F}\cdot d\vec{x}$.



Notice:

- $\vec{x}$ is on sphere $\rho=1$, so $d\vec{x}$ is always $\parallel$ to sphere
- $\vec{F}\parallel\vec{x}$ is always $\perp$ to sphere

$\Longrightarrow \vec{F}\cdot d\vec{x} = 0 \Longrightarrow \int_C \vec{F}\cdot d\vec{x} = 0$.

Coordinate line integrals

Recall $d\vec{x} = \vec{x}'dt = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} dt$

It is reasonable to rewrite this as

$$= \begin{pmatrix} x'\,dt \\ y'\,dt \\ z'\,dt \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

With $\vec{F} = (P,Q,R)$ then, we write

$$\int_C \vec{F}\cdot d\vec{x} = \int_C \begin{pmatrix} P \\ Q \\ R \end{pmatrix}\cdot\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int_C P\,dx + Q\,dy + R\,dz$$

This notation:

- Relates to "differential forms" (Ch. 8). Natural & powerful — but beyond this course!
- In common use.

Symmetry theorems

For scalar line integrals, its just what you would expect:

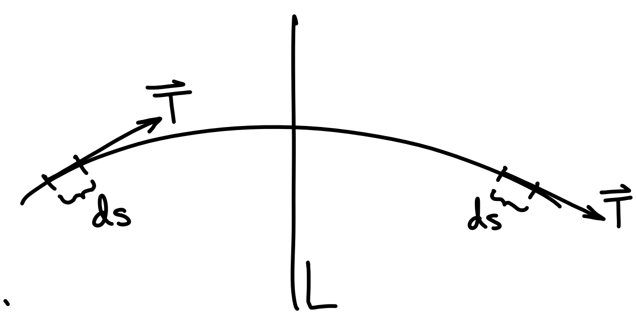
Thm:) If (1) f is odd over L
(2) C is symmetric over L

then $\int_C f\,ds = 0$

NB– Both conditions must hold!
Must use the same line for each!

But for vector line integrals $\int_C \vec{F}\cdot d\vec{x} = \int_C \vec{F}\cdot\vec{T}\,ds$,
odd symmetry does not work!

(On corresponding pieces of curve, the ds's are equal — but the $\vec{T}$'s are neither equal nor opposite...

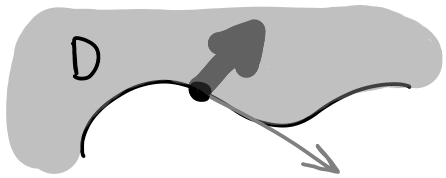


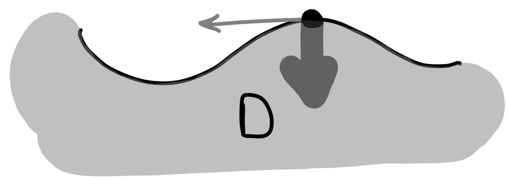
6.2 – Green's Theorem

Boundary

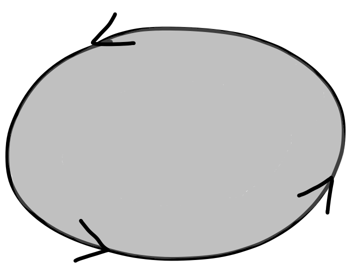
We now view boundary as an oriented thing.

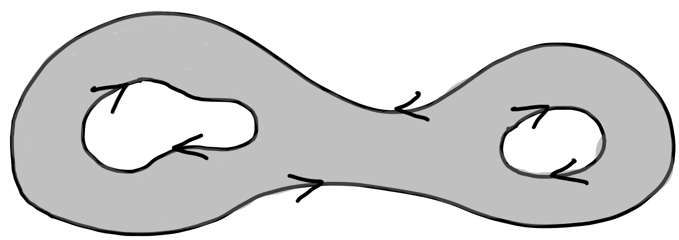
The boundary curve ∂D of a region $D \subset \mathbb{R}^2$ is oriented such that D is "on the left".





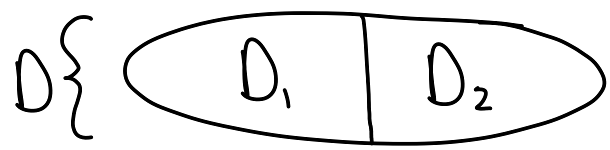
This means that on outside boundary curves the orientation of ∂D is ccw — but not for inside!





Accumulation

Say D is divided into pieces D_1, D_2.



The mass in D is the sum of the masses in D_1, D_2. We say mass "accumulates over D", or is an "accumulating quantity".

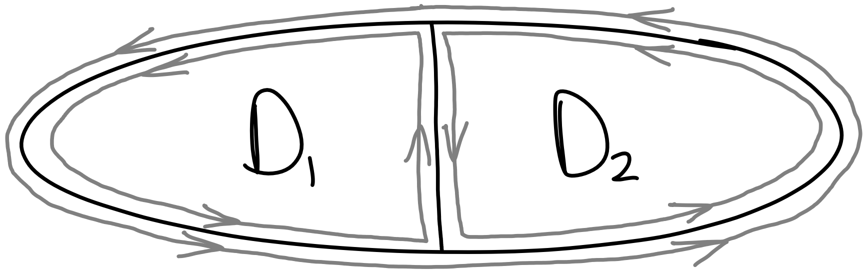
Perimeter does not accumulate over areas!

Tempting to say accumulation is a feature of physical/concrete quantities, not geometric/algebraic ones.

But — given a vector field $\vec{F}$ on D,

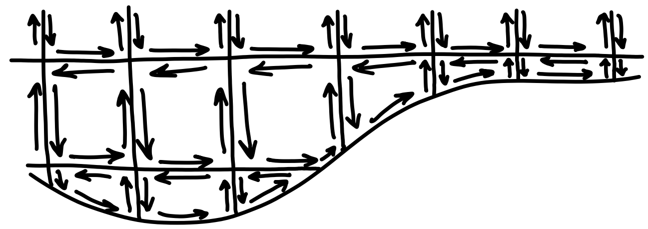
$\boxed{\text{boundary circulation accumulates!}}$

$\int_{\partial D_1}$, $\int_{\partial D_2}$ both go over shared edge, but with opposite orientations which then cancel!

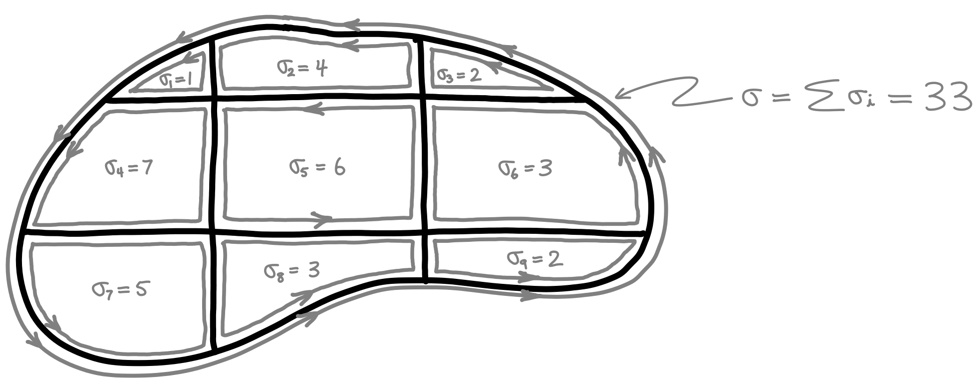


So $$\boxed{\int_{\partial D_1} \vec{F} \cdot d\vec{x} + \int_{\partial D_2} \vec{F} \cdot d\vec{x} = \int_{\partial D} \vec{F} \cdot d\vec{x}}$$

Doesn't matter how many pieces!



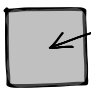
This accumulation feature allows us a way of thinking of boundary circulation as distributed over the interior, with pieces having locations in the interior.



So, "boundary circulation" behaves like a physical "stuff"!

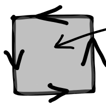
Density

For mass in a differential area, we have "mass density":



$$dm = \left(\frac{\text{mass}}{\text{area}}\right) \cdot dA$$

Similarly, there is a "circulation density"!



$$d\sigma = \frac{\text{bdry. circ.}}{\text{area}} \cdot dA$$

For $\vec{F} = (P, Q)$, circulation density $= \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

(= "Green's operator", $\operatorname{grn} \vec{F}$, $\operatorname{rot} \vec{F}$, $\operatorname{curl} \vec{F}$, $(\nabla \times \vec{F}) \cdot \vec{k}$)

Now consider a domain $D \subset \mathbb{R}^2$ and $\vec{F} = (P, Q)$:

$$\int_{\partial D} \vec{F} \cdot d\vec{x} = \sigma = \iint_D d\sigma = \iint_D (\operatorname{grn} \vec{F})\, dA$$

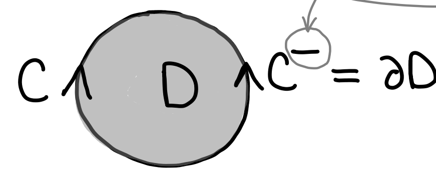
circulation accumulation density

Green's Theorem

Ex:) C is the clockwise unit circle, $\vec{F} = (e^x + y,\ \tan y + 3x)$. Compute $\int_C \vec{F} \cdot d\vec{x}$.

same curve, but with the opposite orientation

Note $C^{-} = \partial D$ where D is the unit disk. Then

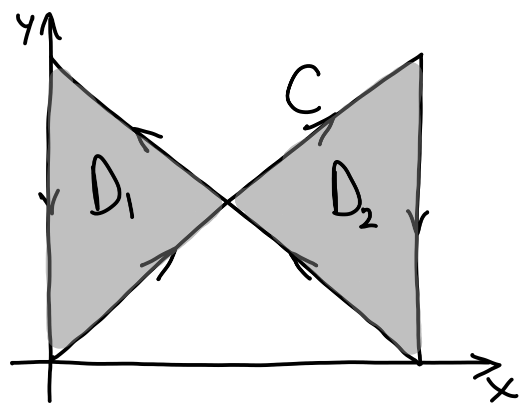


$$\int_C \vec{F} \cdot d\vec{x} = -\int_{C^-} \vec{F} \cdot d\vec{x} = -\int_{\partial D} \vec{F} \cdot d\vec{x} = -\iint_D \operatorname{grn} \vec{F}\, dA$$

$$= -\iint_D (3-1)\, dA = -2\pi$$

Ex:) C is made up of line segments from $(0,0)$ to $(2,2)$ to $(2,0)$ to $(0,2)$ to $(0,0)$, and $\vec{F} = (3x-4y,\ 5x-7y)$. Compute $\int_C \vec{F} \cdot d\vec{x}$.

$$C = \partial D_1 + (\partial D_2)^{-}$$



Then

$$\int_C \vec{F} \cdot d\vec{x} = \int_{\partial D_1} \vec{F} \cdot d\vec{x} + \int_{(\partial D_2)^-} \vec{F} \cdot d\vec{x}$$

$$= \int_{\partial D_1} \vec{F} \cdot d\vec{x} - \int_{\partial D_2} \vec{F} \cdot d\vec{x}$$

$$= \iint_{D_1} \operatorname{grn} \vec{F}\, dA - \iint_{D_2} \operatorname{grn} \vec{F}\, dA$$

$$= \iint_{D_1} 9\, dA - \iint_{D_2} 9\, dA$$

$$= 9(\text{area of } D_1) - 9(\text{area of } D_2) = 0$$

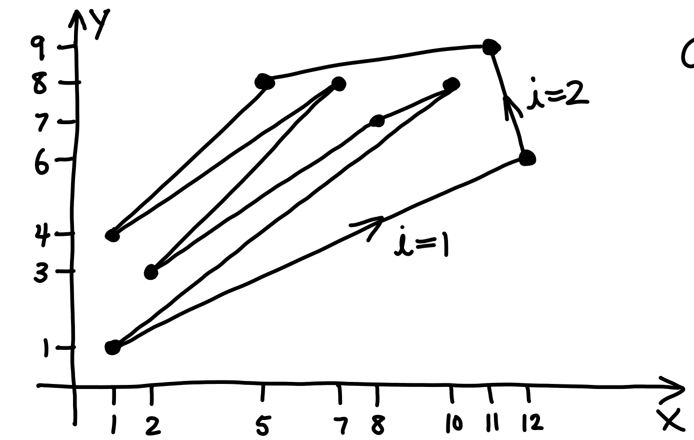
Area

$$\int_{\partial D} x\,dy = \int_{\partial D} \begin{pmatrix}0\\x\end{pmatrix}\cdot d\vec{x} = \iint_D \operatorname{grn}\begin{pmatrix}0\\x\end{pmatrix} dA = \iint_D 1\,dA = \text{area}(D).$$

And for straight line segments (L, from (x_1,y_1) to (x_2,y_2)), you can check that

$$\int_L x\,dy = \bar{x}\,\Delta y \quad \left(=\left(\frac{x_1+x_2}{2}\right)(y_2-y_1)\right)$$

Ex: Compute the area of the polygon below.



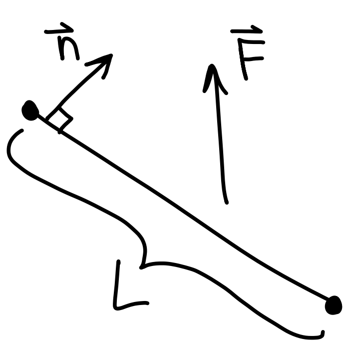
$$\text{area} = \int_{\partial D} x\,dy = \sum_i \int_{L_i} x\,dy$$

i	$\bar{x}$	Δy	$\bar{x}\Delta y = \int_{L_i} x\,dy$
1	$13/2$	5	$65/2$
2	$23/2$	3	$69/2$
3	8	-1	-8
4	3	-4	-12
5	4	4	16
6	$9/2$	-5	$-45/2$
7	5	4	20
8	9	1	9
9	$11/2$	-7	$-77/2$

$\Sigma = \text{area} = 19$

Flux through a curve

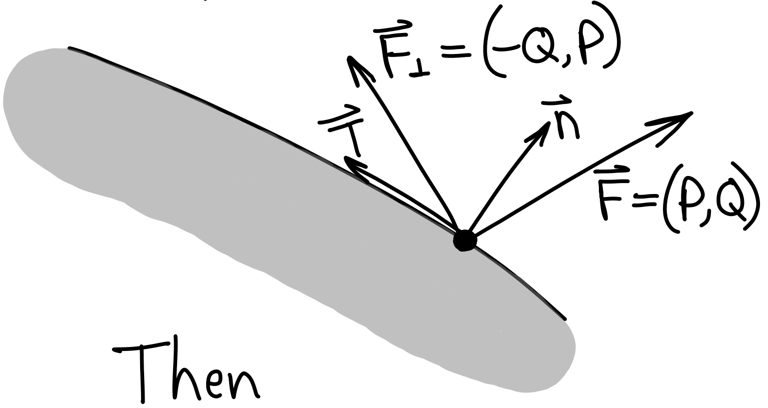
Similar to flux through a surface:



$$\Phi = (\vec{F}\cdot\vec{n})L$$

Represents flow rate of fluid through L (in $\vec{n}$ direction).

"Boundary flux" on D is flow rate <u>outward</u> across ∂D.



$\vec{F}, \vec{n}$ are $90°$ rotations of $\vec{F}_\perp, \vec{T}$ (resp.). So

$$\vec{F}\cdot\vec{n} = \vec{F}_\perp \cdot \vec{T}$$

Then

$$\begin{aligned}
\Phi &= \int_{\partial D} d\Phi \\
&= \int_{\partial D} \vec{F}\cdot\vec{n}\, ds \\
&= \int_{\partial D} \vec{F}_\perp \cdot \vec{T}\, ds \\
&= \int_{\partial D} \vec{F}_\perp \cdot d\vec{x} \\
&= \iint_D \operatorname{grn}(\vec{F}_\perp)\, dA \\
&= \iint_D \frac{\partial(P)}{\partial x} - \frac{\partial(-Q)}{\partial y}\, dA \\
&= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \\
&= \iint_D \nabla\cdot\vec{F}\, dA
\end{aligned}$$

$\boxed{\text{2-d divergence theorem}}$

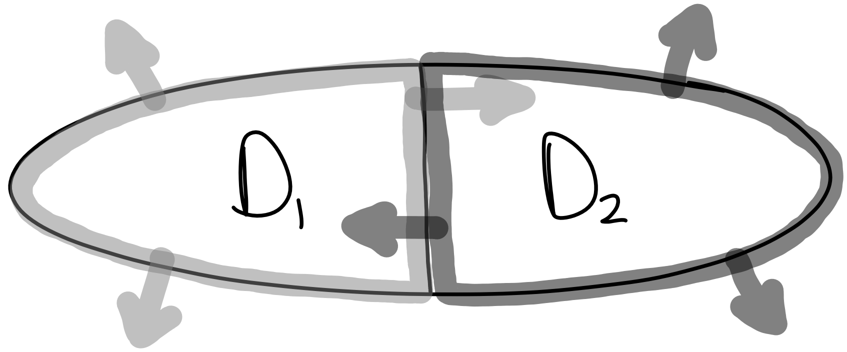
Ex:) Compute the flux $\int_C \vec{F}\cdot\vec{n}\, ds$ of $\vec{F} = (2x + e^y, \csc x - 5y)$ outward through the unit circle C.

C is ∂D where D is the unit disk. So

$$\int_C \vec{F}\cdot\vec{n}\, ds = \Phi = \iint_D \nabla\cdot\vec{F}\, dA = \iint_D (2-5)\, dA = -3(\text{area}) = -3\pi$$

Boundary flux accumulates too!

$\int_{\partial D_1}$, $\int_{\partial D_2}$ both go over shared edge, but with opposite orientations which then cancel!



<u>Alt:</u> on shared edge, any fluid flowing out of D_1 is also flowing into D_2!

$$\text{So} \quad \int_{\partial D_1} \vec{F}\cdot\vec{n}\,ds + \int_{\partial D_2} \vec{F}\cdot\vec{n}\,ds = \int_{\partial D} \vec{F}\cdot\vec{n}\,ds$$

So we have an interpretation as a density as well.

$$m = \iint_D dm = \iint_D \delta \, dA$$
mass density

$$\sigma = \iint_D d\sigma = \iint_D \operatorname{grn}\vec{F} \, dA$$
circulation density

$$\Phi = \iint_D d\Phi = \iint_D \nabla\cdot\vec{F} \, dA$$
flux density

Comparison to F.T.C.

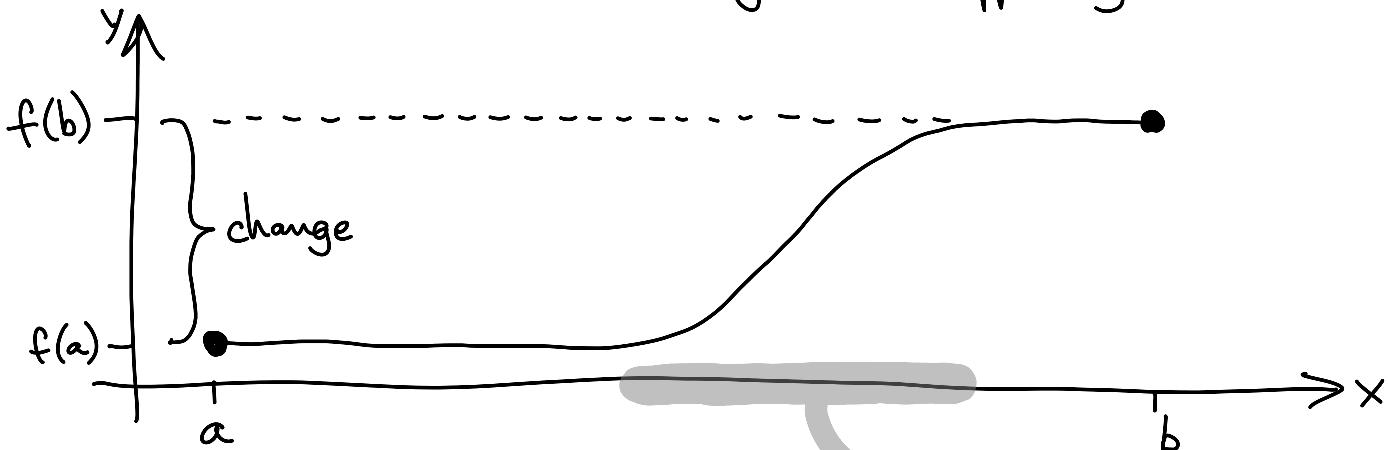
F.T.C. can also be viewed it terms of accumulation/density, viewing $f(x_2)-f(x_1)$ as a quantity called "change" ($=\Delta$). On $I=[a,b]$ then,

change accumulates over I

$$\Delta = \int_I d\Delta = \int_I f' \, dx$$

change/size = change density

The derivative f' is something we can see in a picture, and we know it tells <u>how much</u> change is happening <u>where</u>.

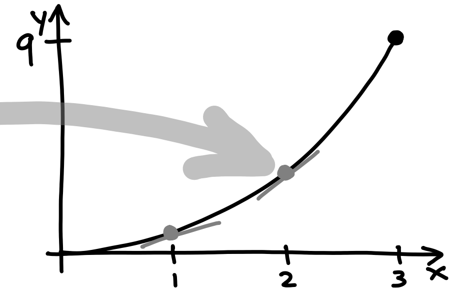


The accumulation of change is <u>measured</u> $\boxed{\text{here}}$... ...but it actually <u>happens</u> $\boxed{\text{here}}$.

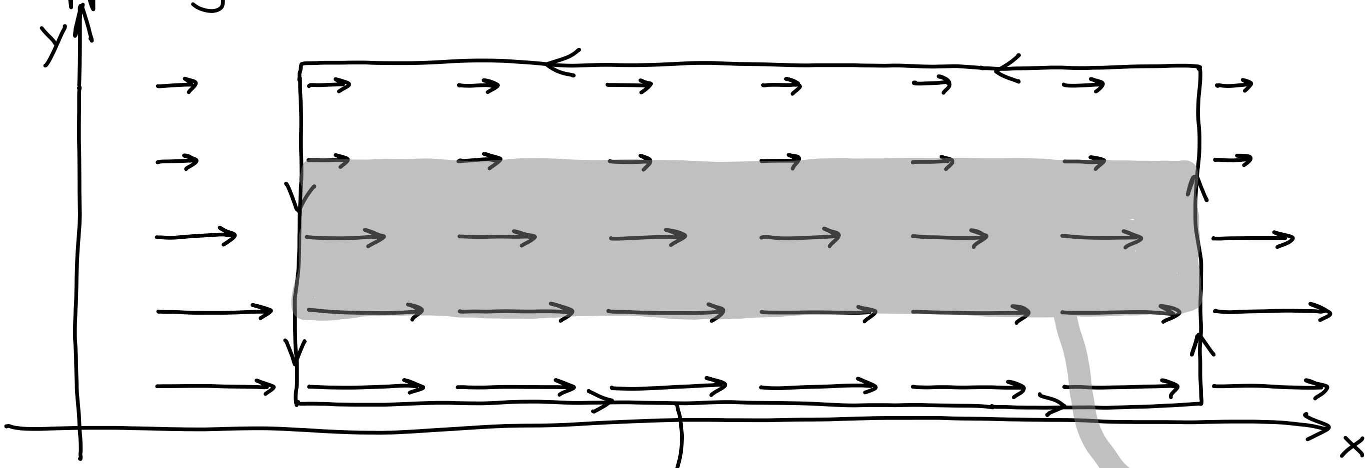
<u>Ex:</u> $f(x)=x^2$ changes by 9 on $[0,3]$. Is more of this change happening at $x=1$ or at $x=2$?

$$f'(x)=2x$$
$$f'(1)=2$$
$$f'(2)=4$$

$\boxed{\text{f}' = \text{change density is greater at } x=2}$

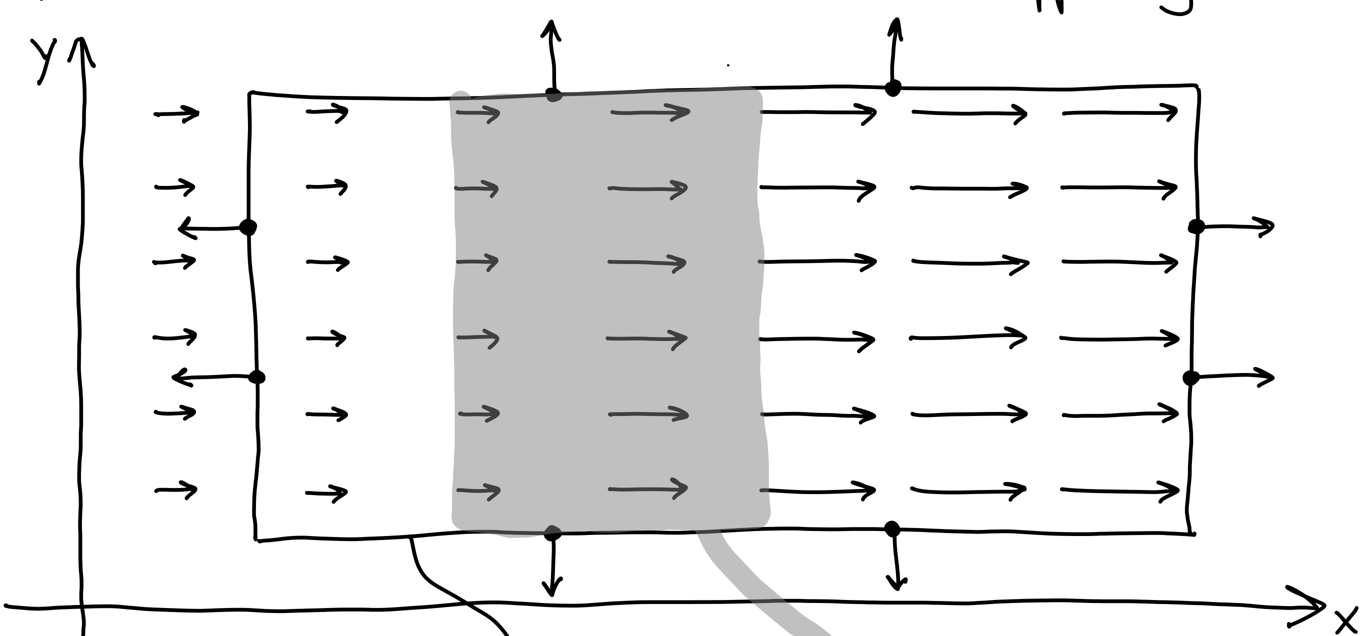


Analogously, grn $\vec{F}$ tells us how much <u>circulation</u> is happening where.



The accumulation of circulation is <u>measured</u> here...   ...but it actually <u>happens</u> here.

And, $\nabla \cdot \vec{F}$ tells us how much <u>flux</u> is happening where.



The accumulation of flux is <u>measured</u> here...   ...but it actually <u>happens</u> here.

Ex: $\vec{F} = (2x^2 - 3y^2, 4x + 5y)$ generates a circulation around, and a flux through, the boundary of $[0,5]\times[1,6]$. For each of these, is more happening at $(3,3)$ or $(2,4)$?

For circulation: $\text{grn}\,\vec{F} = 4 + 6y$

$\text{grn}\,\vec{F}(3,3) = 22$

$\text{grn}\,\vec{F}(2,4) = 28$

$\boxed{\text{grn}\,\vec{F} = \text{circ. dens. is greater at } (2,4).}$

For flux: $\nabla\cdot\vec{F} = 4x + 5$

$\nabla\cdot\vec{F}(3,3) = 17$

$\nabla\cdot\vec{F}(2,4) = 13$

$\boxed{\nabla\cdot\vec{F} = \text{flux density is greater at } (3,3).}$

6.3 – Conservative Vector Fields

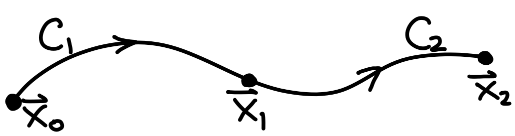
"conservative"

Thm: (Fundamental Theorem of Line Integrals) If $\vec{F} = \nabla f$ is continuous and C starts at A and ends at B, then

$$\int_C \vec{F} \cdot d\vec{x} = f(B) - f(A)$$

This can be seen in terms of accumulation/density!

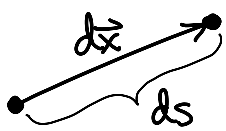
① The accumulating quantity is "change" (Δ), defined on a path P from $\vec{q}$ to $\vec{r}$ as $\Delta = f(\vec{r}) - f(\vec{q})$.



$$\Delta_1 + \Delta_2 = (f(x_1) - f(x_0)) + (f(x_2) - f(x_1))$$
$$= (f(x_2) - f(x_0)) = \Delta$$

<u>Ex:</u> Altitude change accumulates along a road.

② The related density is $(\nabla f \cdot \vec{T})$.



$$d\Delta = \nabla f \cdot d\vec{x} = (\nabla f \cdot \vec{T})\, ds$$

<u>Alt:</u> Can view ∇f as the "oriented density", as it relates $d\Delta$ to the "oriented size" $d\vec{x}$.

Then

$$\Delta = \int_C d\Delta = \int_C (\nabla f \cdot \vec{T})\, ds$$

change density

$$= \int_C \nabla f \cdot d\vec{x}$$

oriented change density

Ex:) $\vec{F} = (2xy^2, 2x^2y)$, $C = \{(\cos t, \sin t) \mid t \in [0,\pi]\}$.
Compute $\int_C \vec{F} \cdot d\vec{x}$.

① Note $\vec{F} = \nabla(x^2y^2)$.
② C starts at $A = (1,0)$ and ends at $B = (-1,0)$.

Then by F.T.L.I. we have

$$\int_C \vec{F} \cdot d\vec{x} = f(B) - f(A) = 0 - 0 = 0$$

Path independence

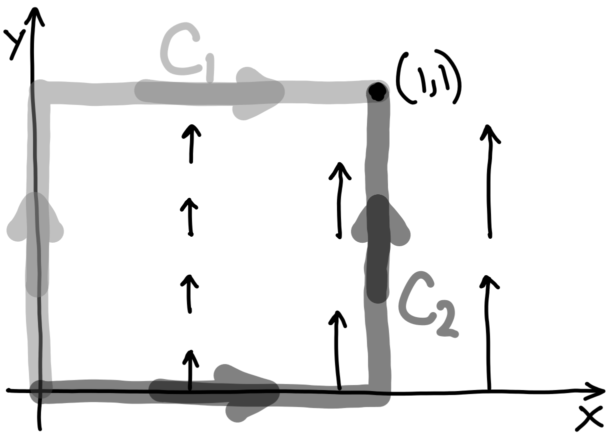
Most line integrals $\int_C \vec{F} \cdot d\vec{x}$ depend heavily on C. Amazingly though, for some vector fields $\vec{F}$, only the endpoints matter!

Def:) $\vec{F}$ "has path independent line integrals" (or, "is path independent") if

$$(C_1, C_2 \text{ share start/end points}) \Rightarrow \left(\int_{C_1} \vec{F} \cdot d\vec{x} = \int_{C_2} \vec{F} \cdot d\vec{x}\right)$$

NB – Most vector fields do not have this property!

Ex: $\vec{F} = (0, x)$, C_1, C_2 as pictured from $(0,0)$ to $(1,1)$.

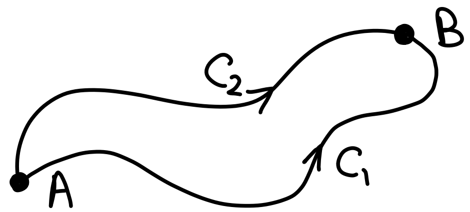


C_1, C_2 start/end at same points.

$\int_{C_1} \vec{F} \cdot d\vec{x}$ is clearly $= 0$.

$\int_{C_2} \vec{F} \cdot dx$ is clearly > 0.

But the F.T.L.I. shows the every (continuous) gradient field $\vec{F} = \nabla f$ is path independent!



$$\int_{C_1} \vec{F} \cdot d\vec{x} = f(B) - f(A) = \int_{C_2} \vec{F} \cdot d\vec{x}$$

It is also true that every (continuous) path independent vector field is a gradient.

Sketch of proof: Can define $f(\vec{x}) = \int_A^B \vec{F} \cdot d\vec{x}$ (use any path!) then check directly that $\nabla f = \vec{F}$.

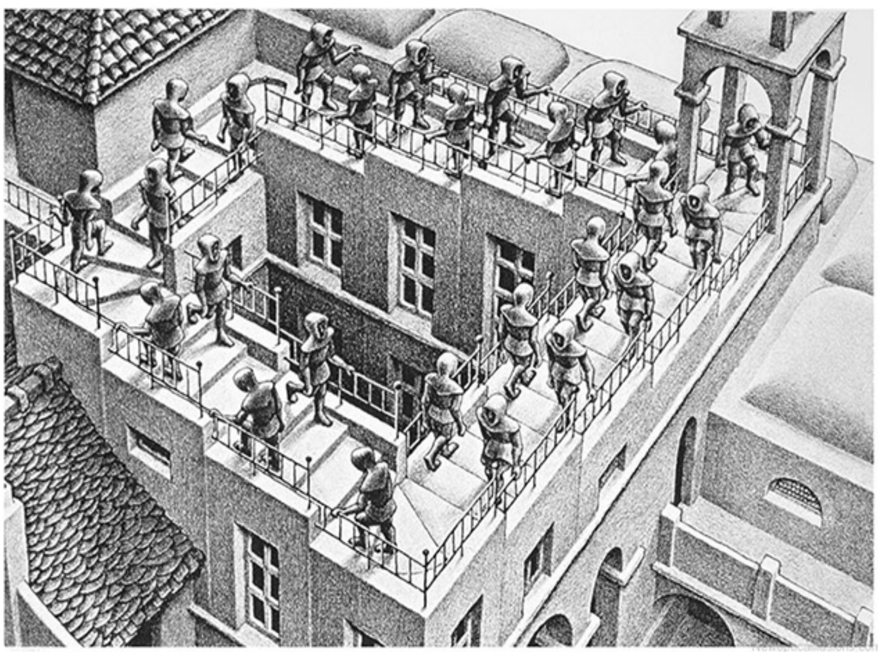
Alterative formulation:

$$\int_{C_1} = \int_{C_2} \iff \int_{C_1} - \int_{C_2} = 0 \iff \int_C = 0$$

$$\left(\vec{F} \text{ is p.i.}\right) \iff \left(\text{for all closed curves, } \oint_C \vec{F} \cdot d\vec{x} = 0\right)$$

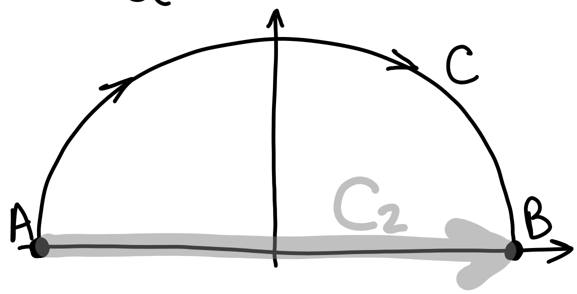
Ex: "Steepness" $= \vec{F} = \nabla h$ is a gradient

$\iff \vec{F}$ is p.i. $\iff \oint_C \vec{F} \cdot d\vec{x} = 0 \iff$ Escher stairs are impossible!



(P.i. is more useful conceptually than computationally, since
(F is p.i.) $\Rightarrow$ ($F = \nabla f$) $\Rightarrow$ (use F.T.L.I.!). But sometimes...

Ex:) $\vec{F} = (2xy, x^2)$ p.i., C is top half of unit circle cwise.
Find $\int_C \vec{F} \cdot d\vec{x}$.



$\vec{F}$ is given p.i., so we can choose to use C_2 instead.

$$\vec{x}(t) = (t, 0)$$

$$\int_C = \int_{C_2} \vec{F} \cdot d\vec{x} = \int_{C_2} \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} \cdot \begin{pmatrix} dx \\ 0 \end{pmatrix} = \int_{C_2} \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \cdot \begin{pmatrix} dx \\ 0 \end{pmatrix} = 0.$$

A local test

Thm:) For a (C^1) vector field $\vec{F}$ on $\mathbb{R}^2$,

$$(\vec{F} = \nabla f) \Longleftrightarrow (\operatorname{grn} \vec{F} = 0)$$

irrotational

Pf:) $\Leftrightarrow$ ($\vec{F}$ p.i.) $\Leftrightarrow$ (closed $\int_C \vec{F} \cdot d\vec{x} = 0$) $\Leftrightarrow$ ($\iint_D \operatorname{grn} \vec{F} \, dA = 0$) $\Leftrightarrow$

Green's Theorem

Thm:) For a (C^1) vector field $\vec{F}$ on $\mathbb{R}^3$,

$$(\vec{F} = \nabla f) \Longleftrightarrow (\nabla \times \vec{F} = \vec{0})$$

Pf:) Similar, but uses a theorem we have not covered yet.

Ex:) Is $\vec{F} = (y+3,\ yz^2+x,\ yz^2+y)$ a gradient?

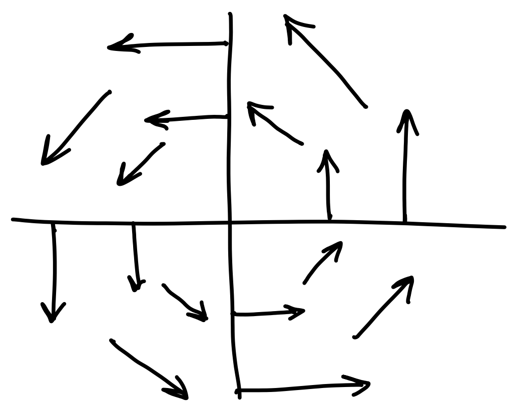
$$\nabla \times \vec{F} = \left((z^2+1) - (2yz),\ \ldots,\ \ldots \right)$$

This already tells us $\nabla \times \vec{F} \neq \vec{0}$, so <u>no</u>.

Ex:) Is $\vec{F} = (-y, x)$ a gradient?

$$\operatorname{grn} \vec{F} = \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} = 2 \neq 0, \text{ so } \underline{\text{no}}.$$

(Alt: If it were, then all of these vectors would be pointing uphill! Escher stairs!)



Strategic diagram for $\mathbb{R}^2$

functions $\xrightarrow{\nabla}$ vector fields $\xrightarrow{\operatorname{grn}}$ functions			$\}$ exact!
points $\xleftarrow{\partial}$ curves $\xleftarrow{\partial}$ regions			$\}$ ~exact!
$\underbrace{f(B) - f(A)}_{\Delta} = \int_C \nabla f \cdot d\vec{x}$	$\underbrace{\int_{\partial D} \vec{F} \cdot d\vec{x}}_{\sigma} = \iint_D \operatorname{grn} \vec{F}\, dA$		$\}$ same pattern!

- Organizes obects, relationships, properties (gradient (conservative) = path independent = irrotational)
- Suggests things to try, what you need!

Ex:) Considering $\int_C \vec{F} \cdot d\vec{x}$...

① Diagram asks if $\vec{F} = \nabla f$ — and suggests computing $\mathrm{grn}\vec{F}$ to decide.

② Diagram asks if $C = \partial D$ — and suggests looking at bdry points to decide.

We will soon see a similar diagram for $\mathbb{R}^3$!

Finding antigradients

If $\vec{F} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ is $\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$, then

$$\begin{aligned} \frac{\partial f}{\partial x} &= P \\ \frac{\partial f}{\partial y} &= Q \\ \frac{\partial f}{\partial z} &= R \end{aligned} \quad \Longrightarrow \quad f = \left\{ \begin{aligned} &\int P\,dx + c_1(y,z) \\ &\int Q\,dy + c_2(x,z) \\ &\int R\,dz + c_3(x,y) \end{aligned} \right\}$$

Need to find f that satisfies all 3 of these requirements!

Ex:) Find an antigradient for $\vec{F} = (2xy+z,\ x^2-z^2,\ x-2yz)$.
(Note that $\nabla \times \vec{F} = \vec{0}$, so we expect such an f to exist.)

$$\int P\,dx = \int 2xy+z\,dx = x^2y + xz \qquad + c_1(y,z)$$
$$\int Q\,dy = \int x^2-z^2\,dy = x^2y \qquad -yz^2 \quad + c_2(x,z)$$
$$\int R\,dz = \int x-2yz\,dz = \qquad xz \ -yz^2 \quad + c_3(x,y)$$

Is there an f equal to all of these? Yes!

$$f = x^2y + xz - yz^2$$

Ex:) Will the above method find an antigradient for $\vec{F} = (-y, x)$? (Note $\operatorname{grn} \vec{F} \neq 0$, so we expect such an $\vec{F}$ should not exist.)

$$\int P\,dx = \int (-y)\,dx = -xy + c_1(y)$$
$$\int Q\,dy = \int (x)\,dy = xy + c_2(x)$$

Is there an f equal to all of these? No!