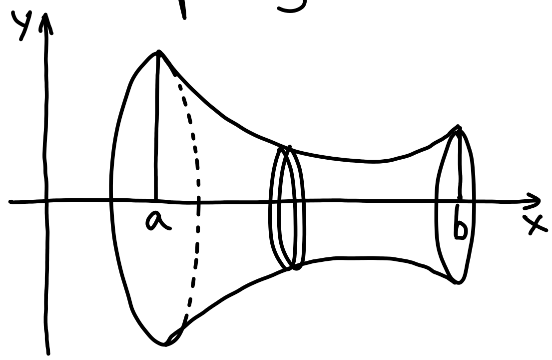
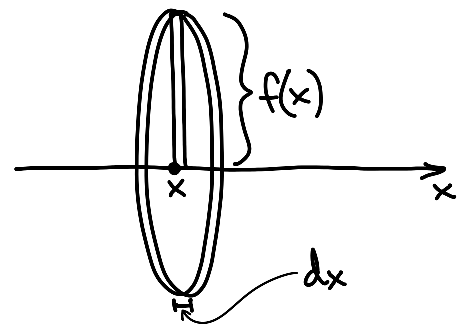
5.1 – Introduction: Areas and Volumes

As with partial derivatives in Chapter 2, we begin with an example that is convenient – but ultimately <u>not</u> the best as a primary point of view.

Recall computing the volume in a rotated surface:

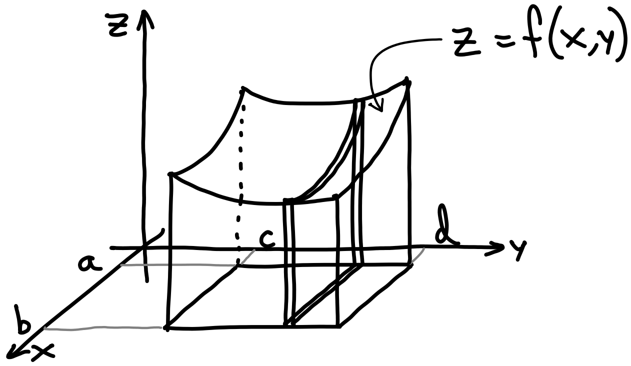


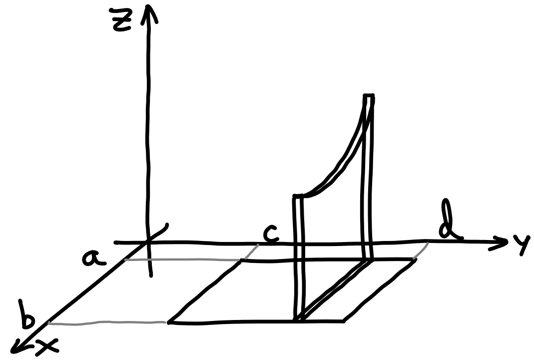


$$V = \int dv \qquad\qquad dv = \pi\left(f(x)\right)^2 dx$$

Conveniently, dv here is a shape we already understand.

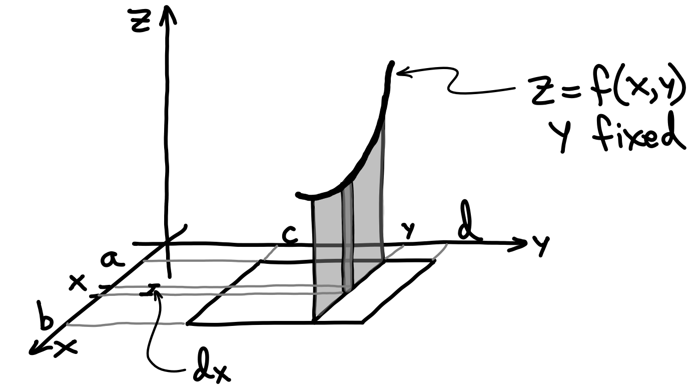
How can we compute the volume under the graph $z = f(x,y)$ (continuous, non-negative) over a rectangle?





$$V = \int dv \qquad\qquad dv = ?$$

One face of the slice is the area under a curve!



So $A(y) = \int_a^b f(x,y)\,dx$

and $dV = A(y)\,dy$

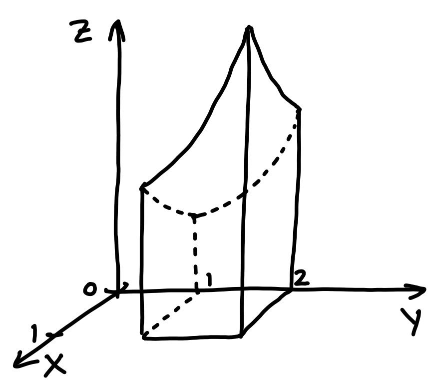
$$= \left(\int_a^b f(x,y)\,dx\right) dy$$

Then $V = \int dV = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$. This is an <u>iterated integral</u>.

NB – dy integral was written 1st, but dx is evaluated 1st!
Use "inside", "outside"!

For the inside integral (dx), the outside variable (y) is <u>constant</u>.

<u>Ex:</u> Find the volume under the graph of $f(x,y) = x^2+y^2$ over the rectangle $[0,1]\times[1,2]$.



$$V = \int_1^2 \left(\int_0^1 x^2+y^2\,dx\right) dy$$

$$= \int_1^2 \left(\frac{1}{3}x^3 + xy^2\right]_{x=0}^{x=1} dy$$

$$= \int_1^2 \left(\left(\frac{1}{3}+y^2\right)-(0)\right) dy$$

$$= \frac{1}{3}y + \frac{1}{3}y^3\Big]_1^2 = \left(\frac{10}{3} - \frac{2}{3}\right) = \frac{8}{3}$$

Alternatively, we could compute this same volume by slicing first in the x-direction. (Think through these details!)

The result is $$V = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$$

So
$$\int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = V = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$$

So you can switch the order of the differentials (and the bounds!).

Warning: This result <u>only</u> works when the domain is a <u>rectangle.</u>

In single variable, we also computed things like the mass of a rod with $\delta = \delta(x)$. What if we want to compute the mass of a rectangular sheet with $\delta = \delta(x,y) \leftarrow$ (mass per unit area) ?

Previously: $$V = \int_c^d \left(\underbrace{\int_a^b f(x,y)\, dx}_{} \right) dy$$ (height)(length) $=$ area

area <u>accumulates</u>

Now...?: $$m \stackrel{?}{=} \int_c^d \left(\underbrace{\int_a^b \delta(x,y)\, dx}_{} \right) dy$$ $\left(\frac{\text{mass}}{\text{area}}\right)$(length) $=$??

<u>accumulates</u>?

This argument has a problem then — for every application other than volume!

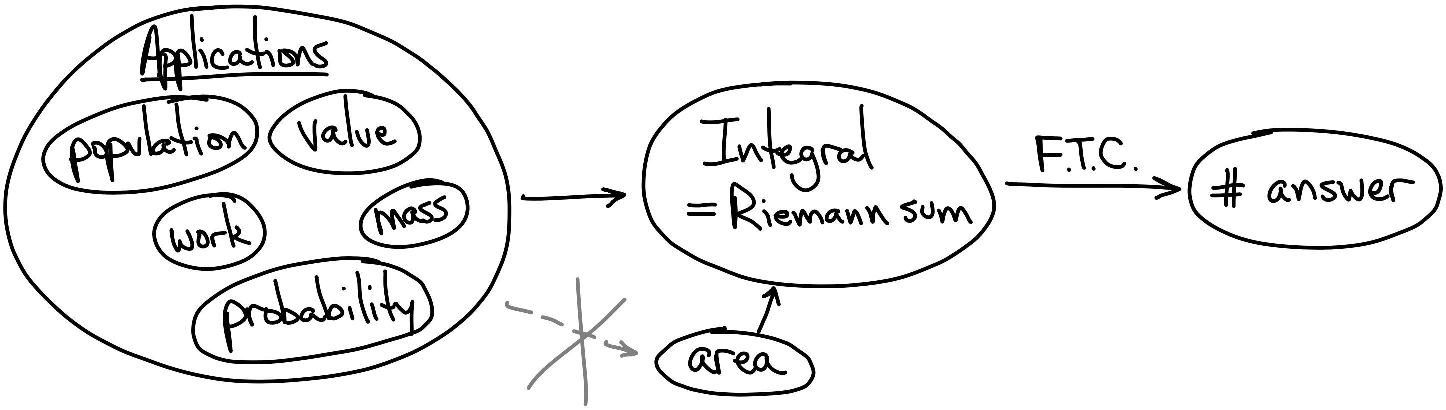
5.2 – Double Integrals

Recall that a single variable integral is a Riemann sum!

Def: For f continuous on $[a,b]$,

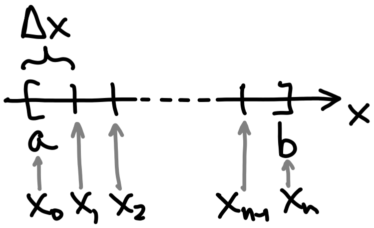
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \left(\Delta x = \frac{b-a}{n},\ x_i = a + i\,\Delta x\right)$$

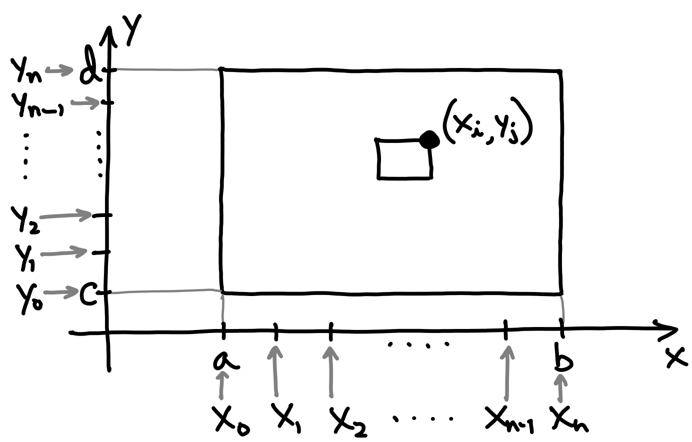
"Area under a graph" is mostly a visual aid — rarely the goal and almost never immediately useful for applications!



Sometimes we need to add up over a rectangle...

We should try to generalize the idea of a Riemann sum to a 2-d domain.





$$\Delta x = \frac{b-a}{n}$$
$$\Delta y = \frac{d-c}{n}$$
$$x_i = a + i\,\Delta x$$
$$y_j = c + j\,\Delta y$$

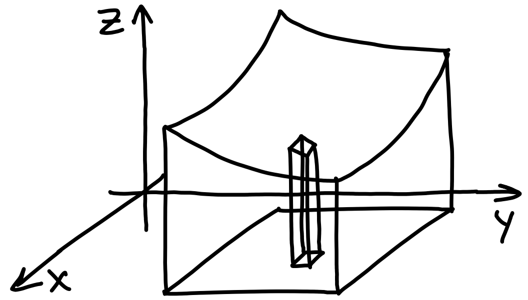
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad\qquad \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y$$

Def: For f continuous on $R = [a,b] \times [c,d]$, the <u>double integral</u> is

$$\iint_R f(x,y)\, dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i, y_j)\, \Delta x\, \Delta y$$

(The book uses "irregular partitions" and "test points"... For continuous $f(x,y)$, these make no difference!)

Ex: Volume under a graph $z = f(x,y)$ (continuous, positive)



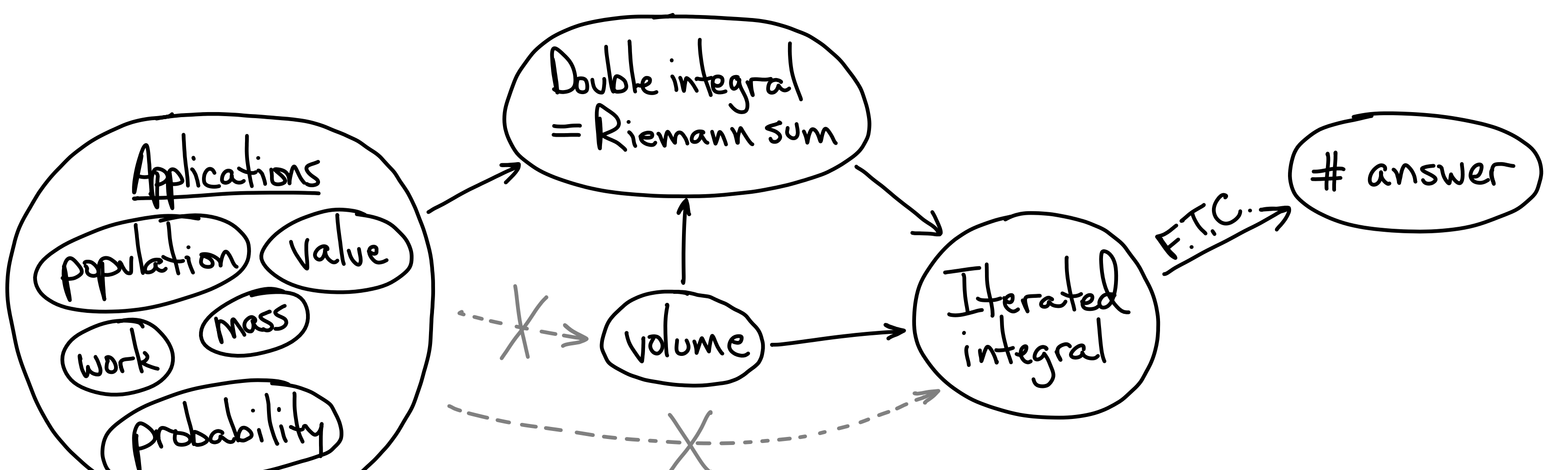
$$\begin{aligned} V &= \lim \sum \Delta V \\ &= \lim \sum f(x_i, y_j)\, \Delta x \Delta y \\ &= \iint_R f(x,y)\, dA \end{aligned}$$

But previously we computed this volume with iterated integrals...

Thm: For f continuous on $R = [a,b] \times [c,d]$,

$$\iint_R f\, dA = \int_a^b \int_c^d f\, dy\, dx = \int_c^d \int_a^b f\, dx\, dy$$

This allows for a 3-step argument for computing:



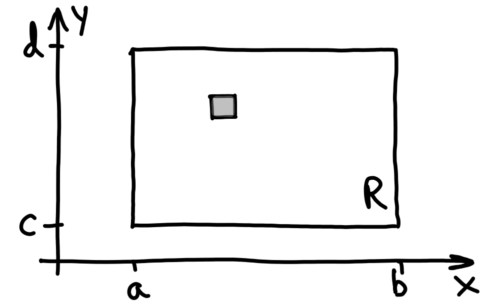
Ex:) Compute the mass on the rectangular sheet $R=[a,b]\times[c,d]$ with $\delta=\delta(x,y)$.

① $m = \lim \sum \Delta m = \lim \sum \delta(x,y)\,\Delta x \Delta y = \iint_R \delta\, dA$

$\left(\text{Alt: } = \iint dm = \iint \delta\, dx\, dy = \iint_R \delta\, dA\right)$

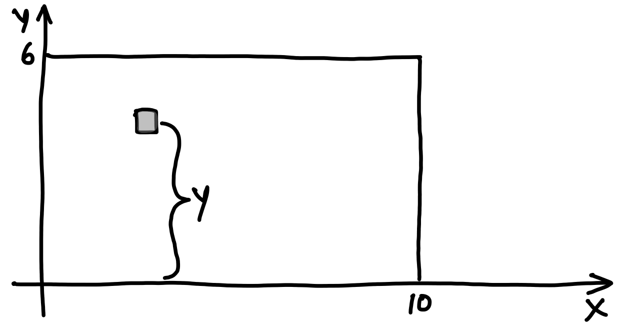
② $\iint_R \delta\, dA = \int_a^b \int_c^d \delta\, dy\, dx$

③ Compute $\int_a^b \int_c^d \delta\, dy\, dx$ with F.T.C.



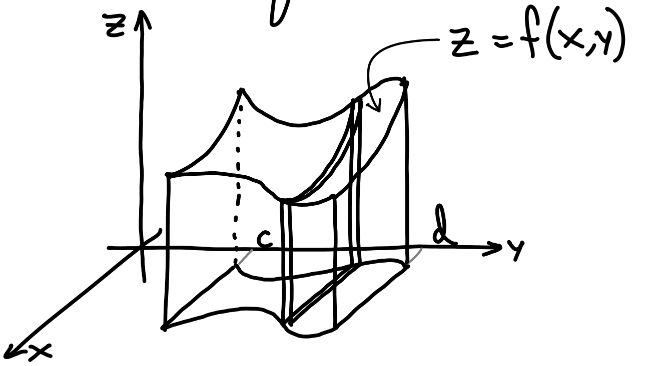
Double integral notation is a convenient shorthand for the development of a Riemann sum.

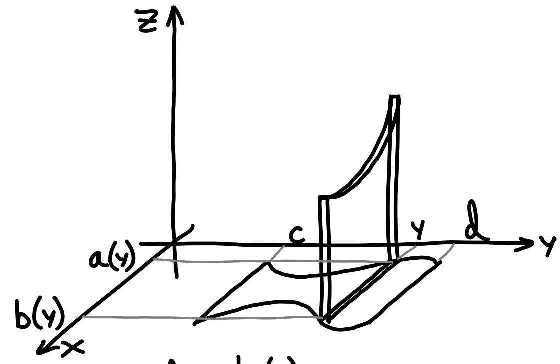
Ex:) A wall is to be $[0,10]\times[0,6]$ in the xy-plane (y is up), with $\delta(x,y) = 1+x+y$. How much work does it take to build this wall using materials on $y=0$?



$$W = \iint dW$$
$$= \iint y\, dF$$
$$= \iint y\,\delta\, dA$$
$$= \int_0^{10}\int_0^6 (y)(1+x+y)\, dy\, dx = \int_0^{10}\int_0^6 y + xy + y^2\, dy\, dx$$
$$= \int_0^{10} \tfrac{1}{2}y^2 + \tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3 \Big]_{y=0}^{y=6} dx = \int_0^{10} 90 + 18x\, dx = 90x + 9x^2\Big]_0^{10}$$
$$= 1800$$

What if the domain is not a rectangle? Think back to the volume question...

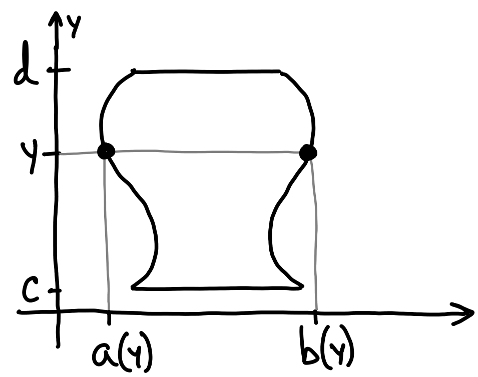




$$V = \int dv \qquad = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

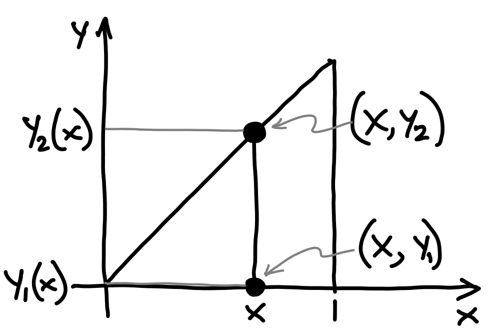
The inner bounds become functions of the outer variable!

Note, you only need to look at the domain to figure out these bounds!



<u>Bounds</u> define <u>points</u> on <u>curves</u> with <u>equations</u>... Solve!

<u>Ex:)</u> D is the triangle with vertices (0,0), (1,0), (1,1). Compute $\iint_D x+y \, dA$.

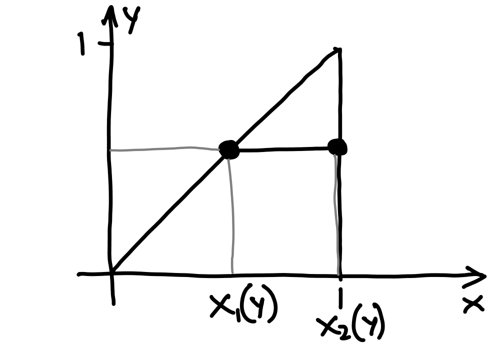


① $x \in [0,1]$

② (x, y_1) is on $y=0$, so $y_1(x) = 0$
(x, y_2) is on $y=x$, so $y_2(x) = x$

$$\iint_D x+y \, dA = \int_0^1 \int_0^x x+y \, dy\, dx = \int_0^1 \left(xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=x} dx = \int_0^1 \tfrac{3}{2}x^2\, dx \ldots$$

Alt:



① $y \in [0,1]$

② (x_1, y) is on $y = x$, so $x_1(y) = y$

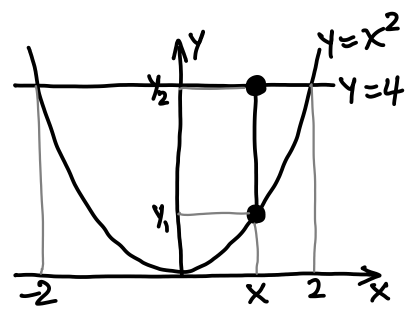
(x_2, y) is on $x = 1$, so $x_2(y) = 1$

$$\iint_D x+y \, dA = \int_0^1 \int_y^1 x+y \, dx \, dy = \int_0^1 \left(\tfrac{1}{2}x^2 + xy\right]_{x=y}^{x=1} dy$$

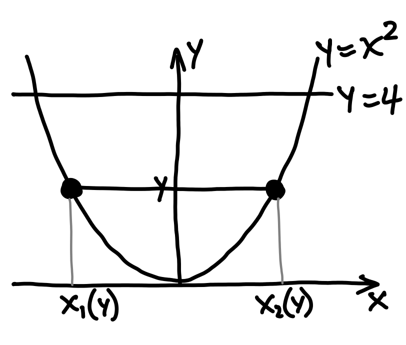
$$= \int_0^1 \left(\tfrac{1}{2} + y\right) - \left(\tfrac{3}{2}y^2\right) dy = \ldots$$

Ex:) D is bounded by $y = x^2$ and $y = 4$. Write $\iint_D f \, dA$ as an iterated integral.

$$\int_{-2}^{2} \int_{x^2}^{4} f \, dy \, dx$$

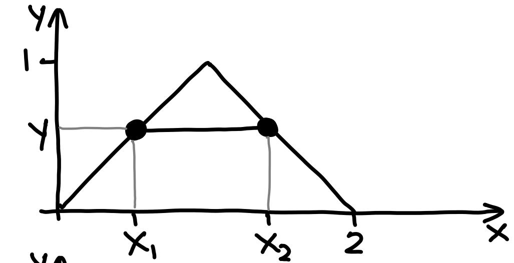


$$\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} f \, dx \, dy$$

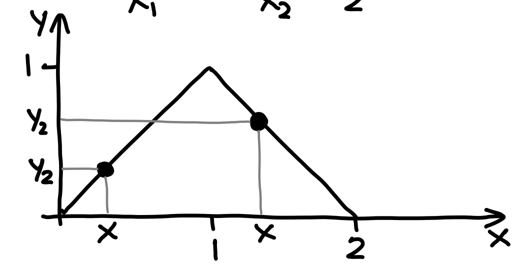


All else being equal, if the bound equation(s) have a clear "input" variable, its nice to put that on the outside.

Ex:) D is the triangle with corners $(0,0)$, $(1,1)$, $(2,0)$.
Write $\iint_D f\,dA$ as an iterated integral.



$$\int_0^1 \int_y^{2-y} f \; dx\,dy$$



$$\int_0^1 \int_0^x f \, dy\,dx + \int_1^2 \int_0^{2-x} f \, dy\,dx$$

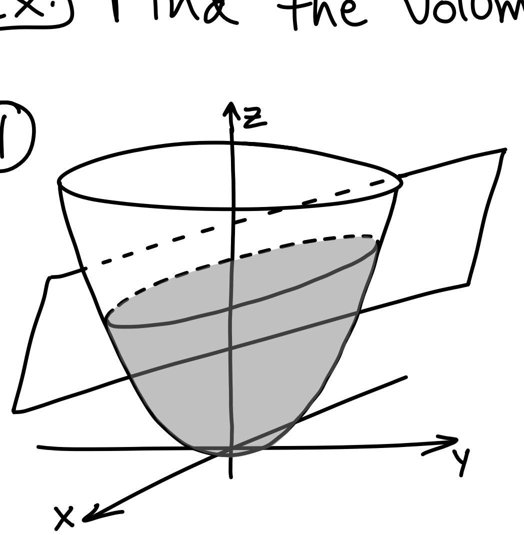
If you can, try to slice so that your inner bound functions are nice!

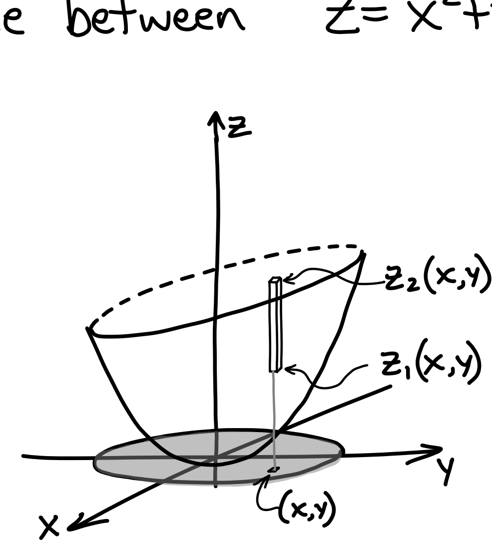
Alt.: Try not to slice through corners!

Sometimes you have to find the domain.

Ex:) Find the volume between $z = x^2+y^2$, $4x+8y+2z=7$.

(1)





$$V = \iint dV$$

$$= \iint (z_2 - z_1)\, dA$$

(2) Domain is determined by where these surfaces intersect — where $z_1(x,y) = z_2(x,y)$!

$$x^2 + y^2 = \frac{7}{2} - 2x - 4y$$

$$x^2+2x + y^2+4y = \frac{7}{2}$$

$$(x^2+2x+1) + (y^2+4y+4) = \frac{7}{2} + 1 + 4$$

$$(x+1)^2 + (y+2)^2 = \frac{17}{2}$$

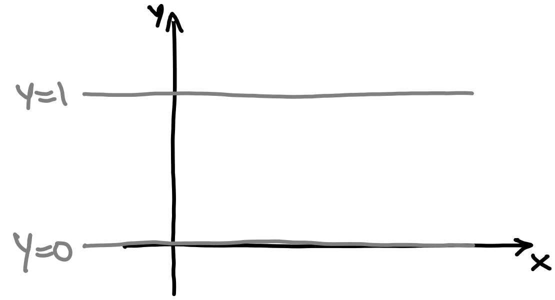
5.3 – Changing the Order of Integration

From an iterated integral you can infer the domain of the double integral, and then slice in the other order. Sometimes this is useful.

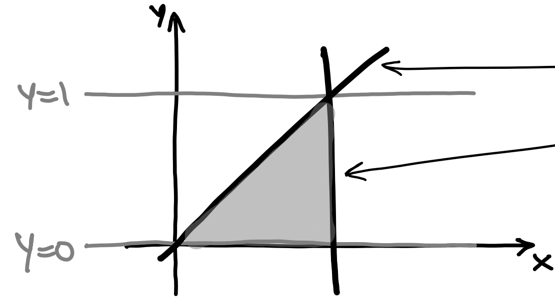
Ex:) Compute $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$

(Problem: $\int e^{x^2}\,dx$ can't be written in closed form...)

The outer bounds mean the domain D is bounded by the lines $y=0$, $y=1$:



The inner bounds mean D is bounded by $x(y)=y$, $x(y)=1$:



So the shaded region is D. Slicing the other way gives

$$\iint_D e^{x^2}\,dA = \int_0^1 \underbrace{\int_0^x e^{x^2}\,dy}\,dx$$

$$= \int_0^1 x e^{x^2}\,dx$$

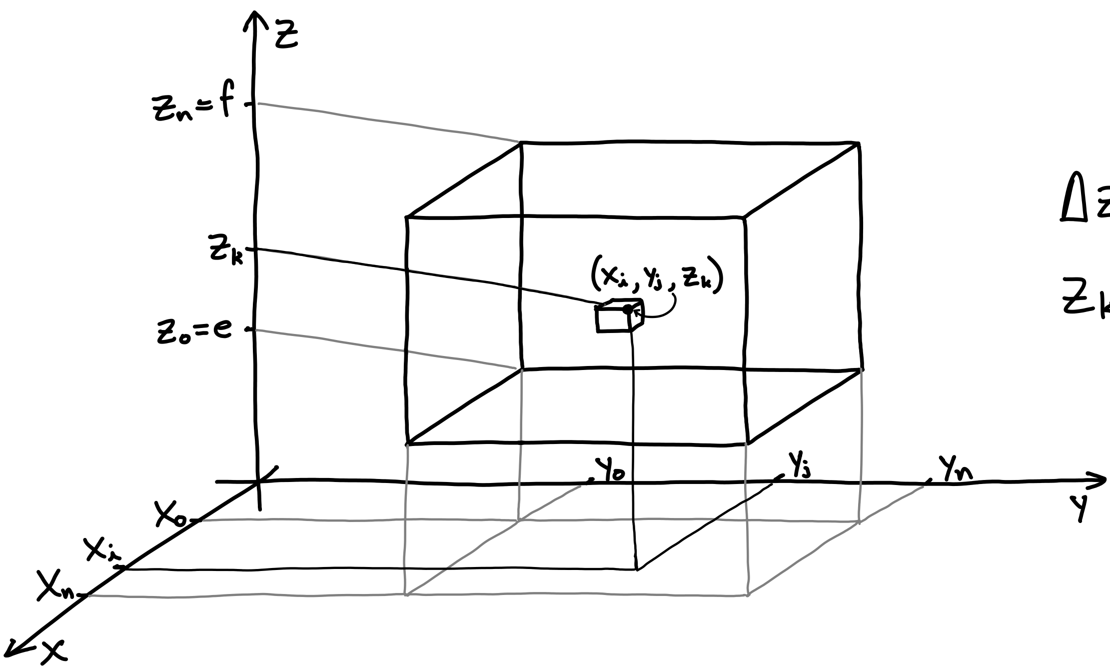
which can be computed by the substitution $u=x^2$!

5.4 - Triple Integrals

Sometimes we need to add up over a 3-d region!

- How much mass is in a solid with $\delta = \delta(x,y,z)$?
- What is the probability of an event with p.d.f. $f(x,y,z)$?

We should try to generalize the idea of a Riemann sum to a 3-d domain.



$$\Delta z = \frac{f-e}{n}$$

$$z_k = e + k\Delta z$$

Def:) For f continuous on $R = [a,b] \times [c,d] \times [e,f]$, the triple integral is

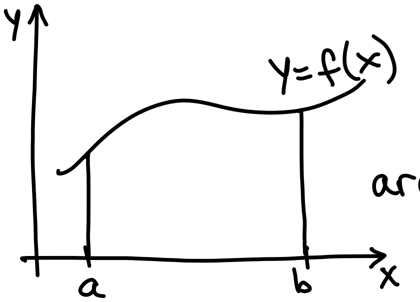
$$\iiint_R f(x,y,z)\, dV = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x_i, y_j, z_k)\, \Delta x \, \Delta y \, \Delta z$$

(The book uses "irregular partitions" and "test points"... For continuous $f(x,y)$, these make no difference!)

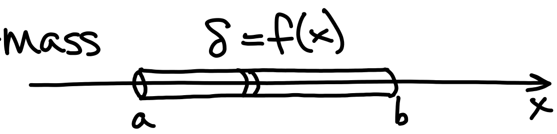
Ex:) What is the mass in R with $\delta=\delta(x,y,z)$?

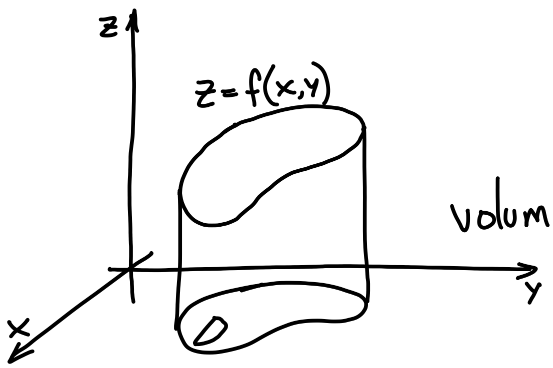
$$m=\iiint dm = \iiint_R \delta(x,y,z)\,dV$$

There is no "standard" geometric example/application...

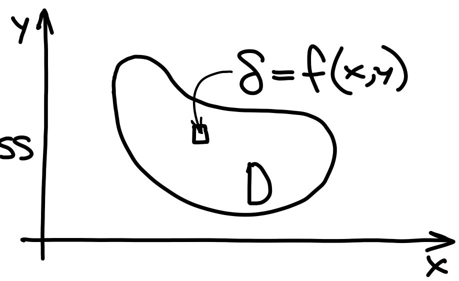


area = $\int_a^b f(x)\,dx$ = mass

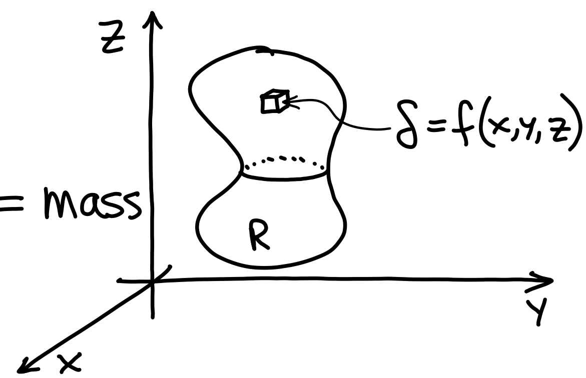




volume = $\iint_D f(x,y)\,dA$ = mass



??? = $\iiint_R f(x,y,z)\,dV$ = mass



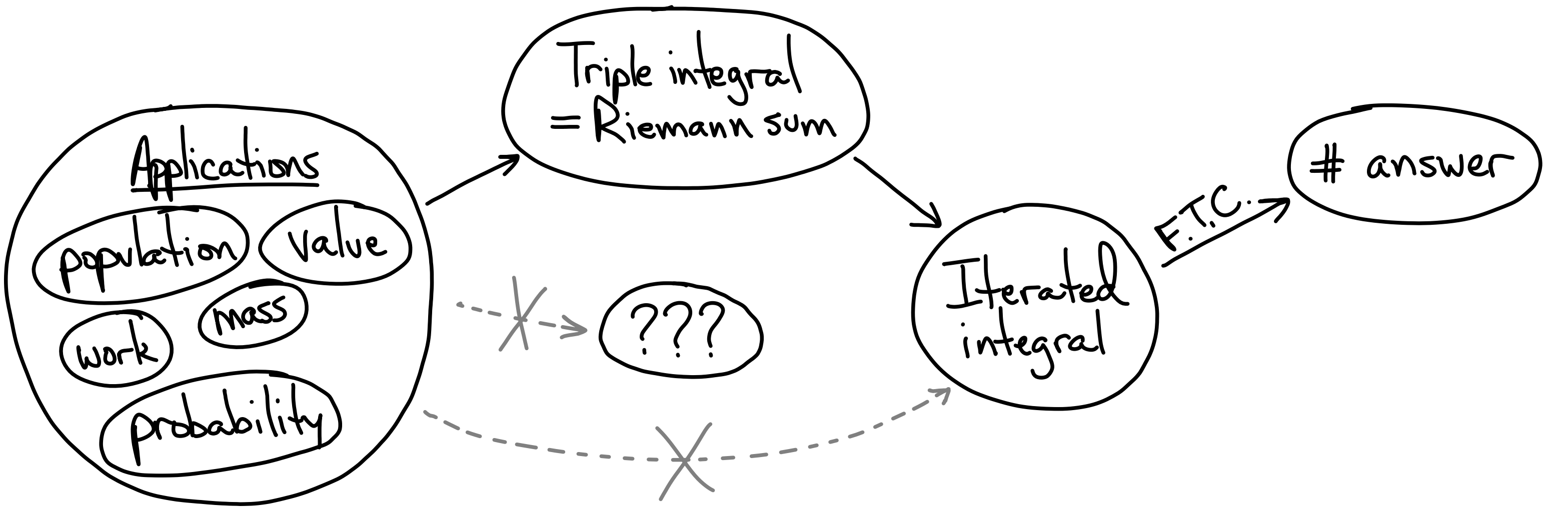
This is another argument against the graph point of view on single/double integrals... It sets up an unsustainable pattern!

Without the graph point of view we can't argue like we did for double integrals. But still:

Thm:) For f continuous on $R = [a,b] \times [c,d] \times [e,f]$,

$$\iiint_R f \, dV = \int_a^b \int_c^d \int_e^f f \, dz \, dy \, dx = \int_e^f \int_c^d \int_a^b f \, dx \, dy \, dz = \cdots$$

This allows for a 3-step argument for computing:



Ex:) Compute the mass in $R = [1,2] \times [0,2] \times [0,1]$ with $\delta(x,y,z) = 1 + 2xyz$.

$$m = \iiint_R \delta \, dV = \int_0^2 \int_1^2 \int_0^1 1 + 2xyz \, dz \, dx \, dy$$

$$= \int_0^2 \int_1^2 \left(z + xyz^2 \right]_{z=0}^{z=1} dx \, dy$$

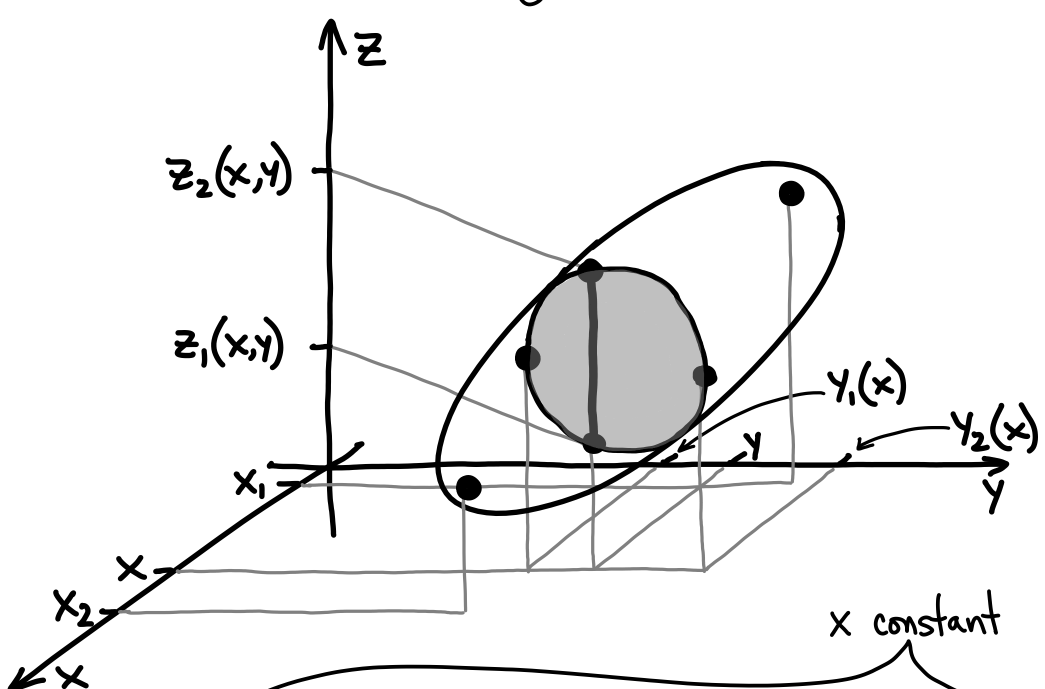
$$= \int_0^2 \int_1^2 1 + xy \, dx \, dy = \int_0^2 \left(x + \tfrac{1}{2} x^2 y \right]_{x=1}^{x=2} dy$$

$$= \int_0^2 1 + \tfrac{3}{2} y \, dy = \left(y + \tfrac{3}{4} y^2 \right]_0^2 = 5$$

For non-rectangular domains, again we can't make a geometric argument like we did for double integrals.

But the process is analogous, constructing the iterated integral:

1. Slice it orthogonal to a coordinate axis.
2. Identify the bounds for that coordinate integral by the locations of the first and last slices.
3. Fix a slice, by a generic value of that variable.



$$\int_{x_1}^{x_2} \overbrace{\int_{y_1(x)}^{y_2(x)} \overbrace{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz}^{x,y \text{ constant}}\, dy}^{x \text{ constant}}\, dx$$

range of x over D

range of y over x-slice

range of z over y-slice of x-slice

"inside"

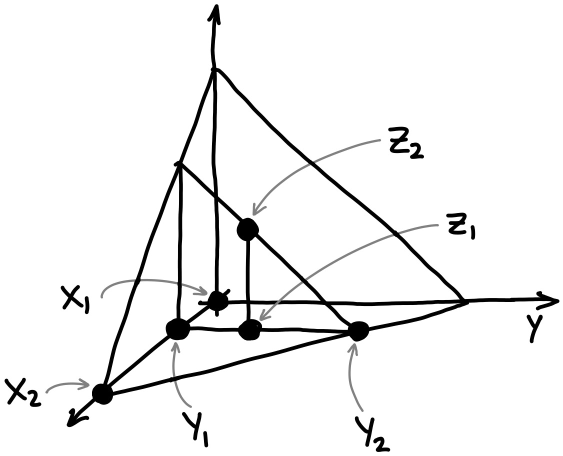
"middle" or "second"

"outside"

Note:

- This constructs the iterated integral <u>"outside" to "inside"</u>.
- Each bound can be a function of variable(s) further to the outside.
- <u>Bounds</u> come from <u>points</u> on <u>surfaces</u> with <u>equations</u>.

<u>Ex:</u>) T is the tetrahedron bounded by
$x=0$, $y=0$, $z=0$, $x+y+z=1$.



x_2: $y=0$, $z=0$, $x+y+z=1$ $\Rightarrow x_2=1$
y_2: $x=x$, $z=0$, $x+y+z=1$ $\Rightarrow y_2=1-x$
z_2: $x=x$, $y=y$, $x+y+z=1$ $\Rightarrow z_2=1-x-y$

$$\iiint_T f\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} f\,dz\,dy\,dx$$

You can slice in any of 6 orders. Choose wisely!

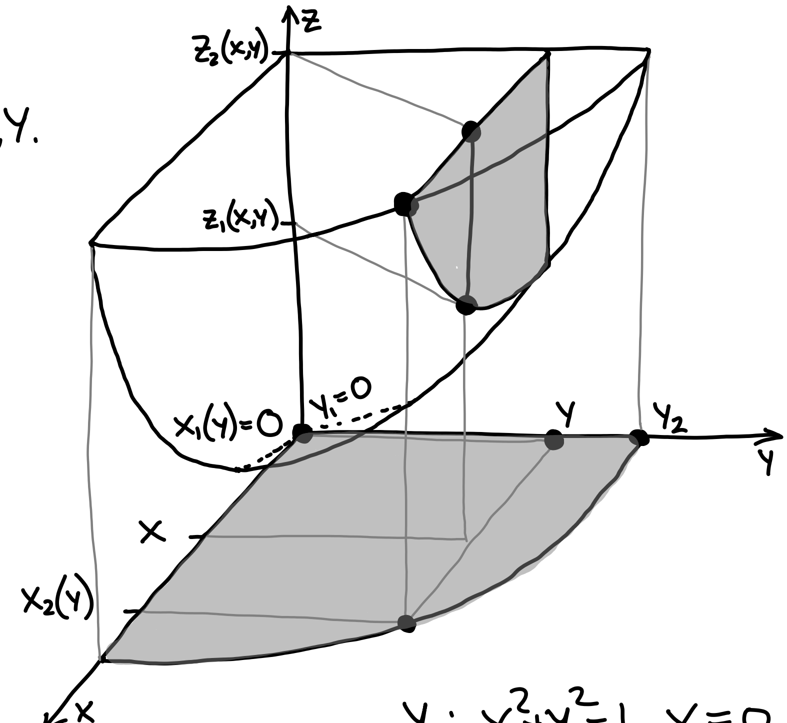
- Try not to slice through corners.
- If one variable is more naturally a function of the others, put it on the inside.
- Look for "good" projections:
  - "shadow" is desirable;
  - "top", "bottom" surfaces each consistent & desirable;
  - then, with those two variables on the outside, the projection gives the bounds!

Ex:) R is the region in the first octant bounded by $z=1$, $z=x^2+y^2$.

① z is best as a fn of x,y.

② "good" projection to xy-plane.

So we choose "$dz\,dx\,dy$" and look at the xy shadow for the x & y bounds.



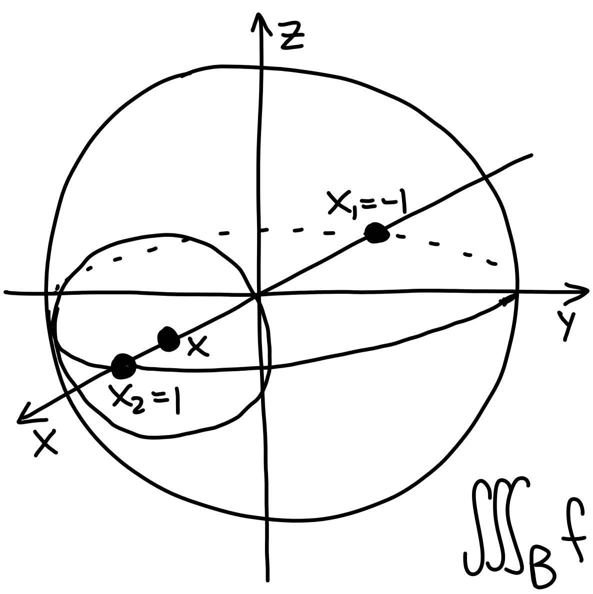
Circular edge is where $z=x^2+y^2$, $z=1$, so $x^2+y^2=1$.

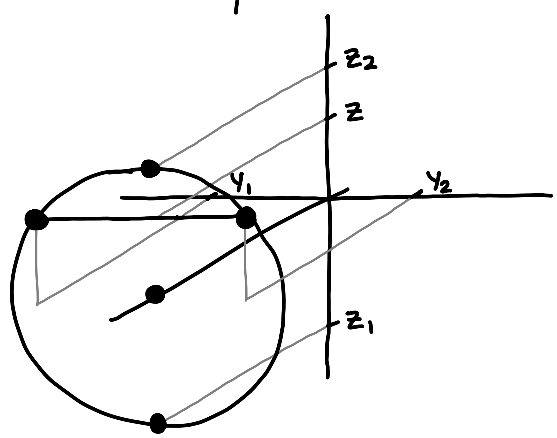
y_2: $x^2+y^2=1$, $x=0$
$\Rightarrow y_2=1$

$$\iiint_R f\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{1} f \; dz\,dx\,dy$$

x_2: $y=y$, $x^2+y^2=1$
$\Rightarrow x_2=\sqrt{1-y^2}$

Ex:) B is the unit ball, choose $dy\,dz\,dx$.

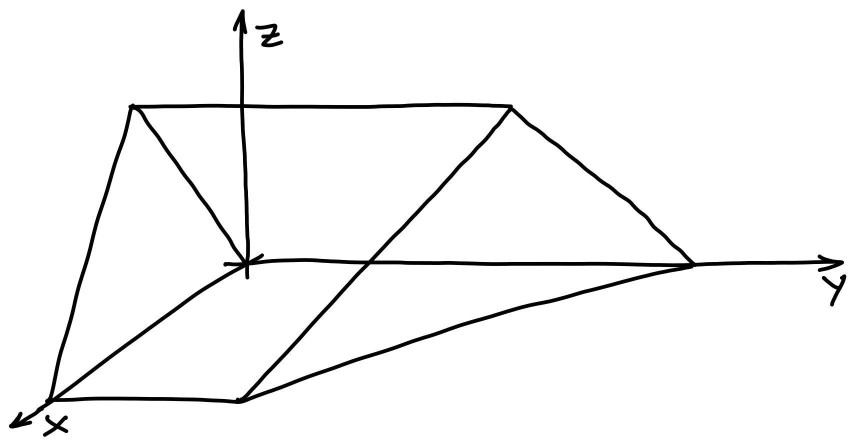




$$\iiint_B f\,dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f \; dy\,dz\,dx$$

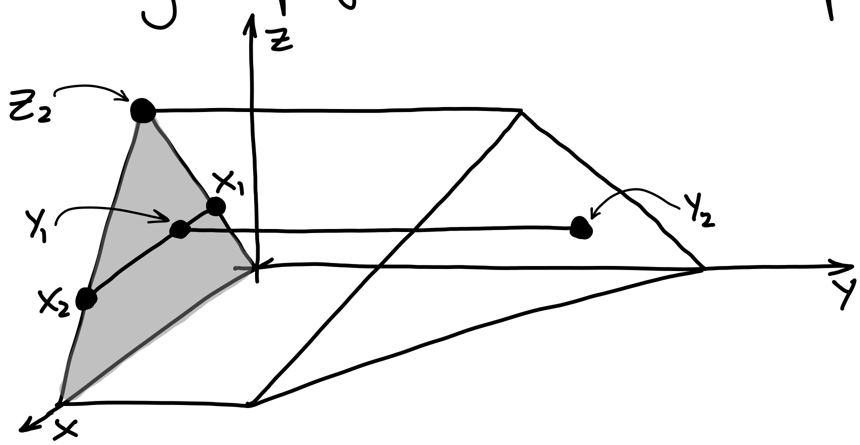
Ex:) R is bounded by: $y=0$, $z=0$, $z=x$, $x+y+z=3$, $x+z=2$.

① Trick for drawing: Note 3 eq's have no y's $\Rightarrow$ $\parallel$ to y-axis.



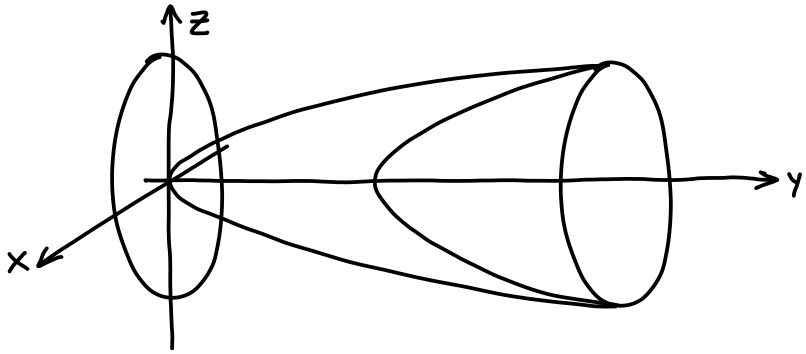
②a) Note that the only slicing that avoids corners is $dy\,dx\,dz$.

②b) There is a good projection to the xz-plane.



③
$$\iiint_R f\,dV = \int_0^1 \int_z^{2-z} \int_0^{3-x-z} f \; dy\,dx\,dz$$

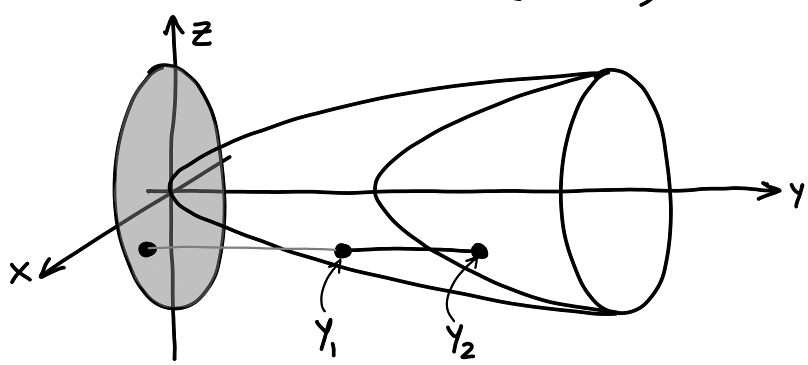
Ex:) R is bounded by $y = x^2+z^2+4$, $y = 2(x^2+z^2)$.



ⓐ y is dependent
ⓑ good proj. to xz-plane
$\Rightarrow$ choose $dy\,dz\,dx$

Shadow is defined by the intersection:

$$y_1 = y_2 \implies x^2+z^2+4 = 2(x^2+z^2) \implies x^2+z^2 = 4$$



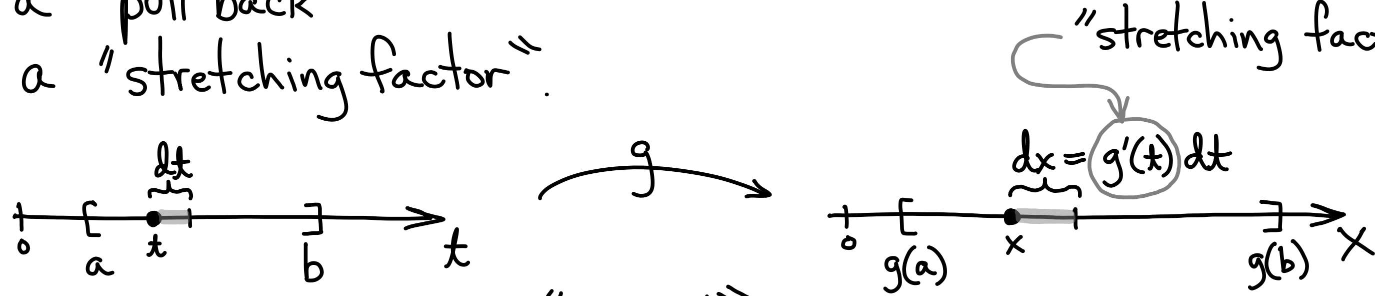
So $$\iiint_R f\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2(x^2+z^2)}^{x^2+z^2+4} f\,dy\,dz\,dx$$

5.5 - Change of Variables

The substitution rule can be thought of in terms of

① a "pull back"

② a "stretching factor".

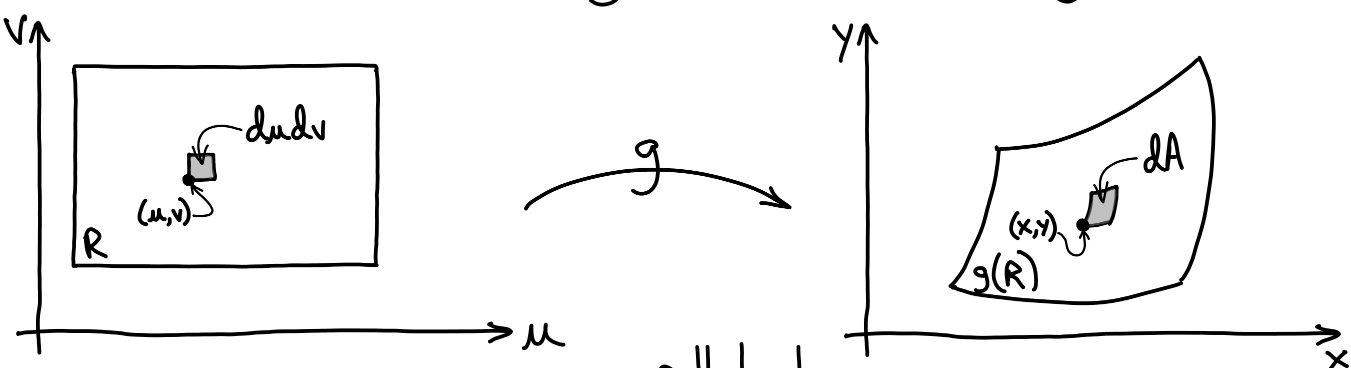


"pull back"

$$\int_a^b f(g(t))\; g'(t)\, dt \;=\; \int_{g(a)}^{g(b)} f(x)\; dx$$

This changes both the <u>integrand</u> and the <u>domain.</u>

We can do the same thing for double integrals.



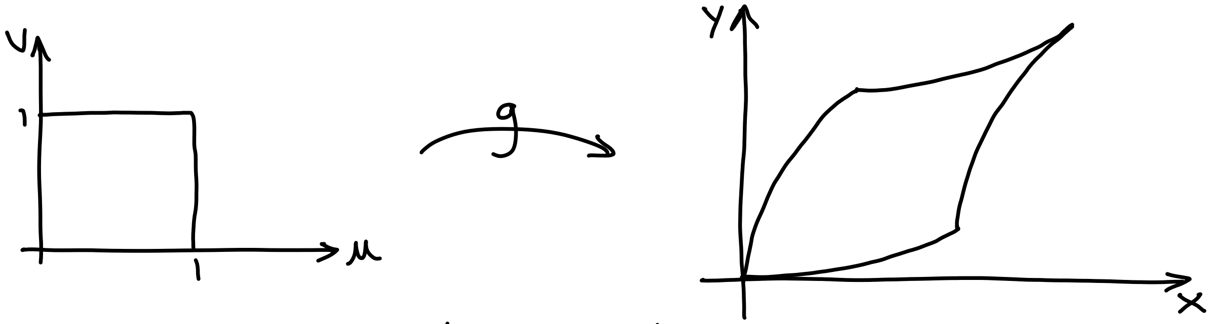
pull back

$$\iint_R f(g(u,v))\; (\text{s.f.})\, du\, dv \;=\; \iint_{g(R)} f(x,y)\; dA$$

Turns out the stretching factor is

$$(\text{s.f.}) = \left|\det Dg\right| = \left|\det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

Ex:) Compute $\iint_D x+y \, dA$, where D in the xy-plane is the image of the unit square S in the uv-plane by $g(u,v) = (u^2+3v, v^2+3u) = (x,y)$.



$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2u & 3 \\ 3 & 2v \end{pmatrix} = 4uv - 9$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |4uv-9| = 9-4uv$$

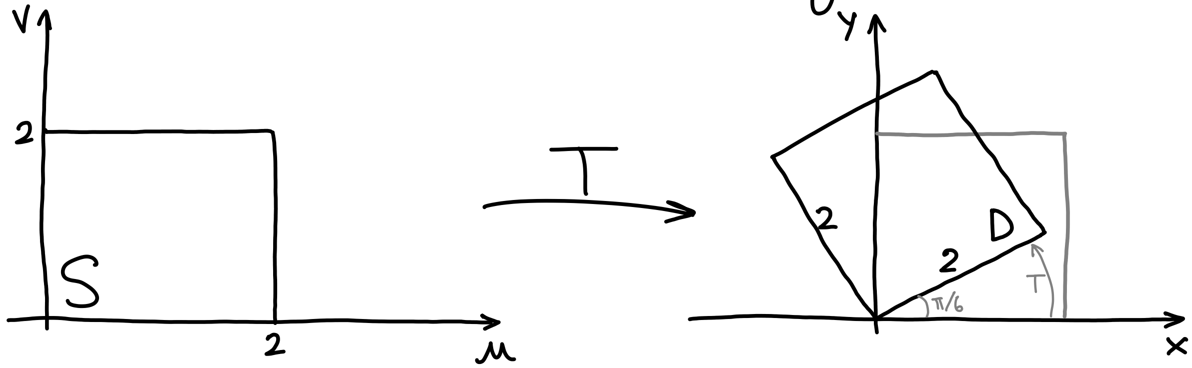
Then $\iint_D x+y \, dA = \iint_S \left((u^2+3v)+(v^2+3u)\right)(9-4uv) \, du \, dv$

Some useful transformations:

This g...	... does this in the pull back:
$g(u,v) = (u+a, v+b)$	translates by $(-a,-b)$
$g(u,v) = (ku, v)$	squishes by k in x-direction
g = ccw rotation by θ	rotates cw by θ

Ex: D is the square with corners at $\vec{0}$, $(\sqrt{3},1)$, $(-1,\sqrt{3})$, $(\sqrt{3}-1,\sqrt{3}+1)$.
Compute $\iint_D 2x-y \, dA$.

D is a ccw rotation of an easier square:



$$T\begin{pmatrix}u\\v\end{pmatrix} = A\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$$\det DT = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix} = 1$$

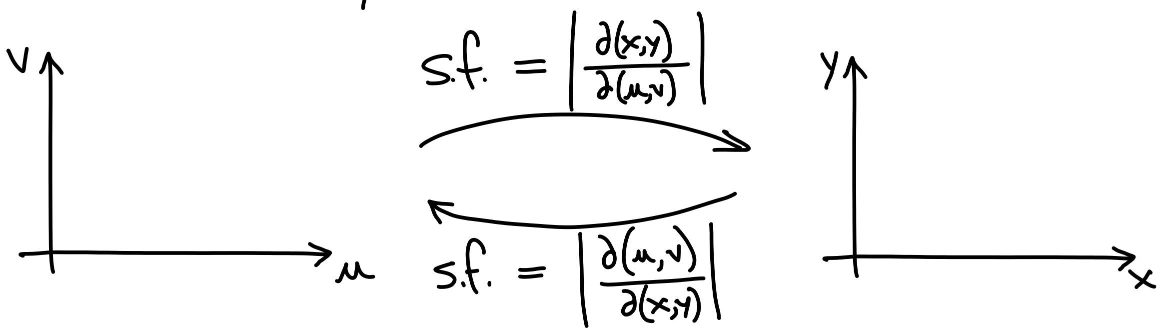
(This is geometrically expected — but, show the calculation!)

Then
$$\iint_D 2x-y \, dA = \iint_S \left(2\left(\tfrac{\sqrt{3}}{2}u - \tfrac{1}{2}v\right) - \left(\tfrac{1}{2}u + \tfrac{\sqrt{3}}{2}v\right)\right)\left(|1| \, du \, dv\right)$$
$$= \int_0^2\int_0^2 \left(\sqrt{3}-\tfrac{1}{2}\right)u + \left(-1-\tfrac{\sqrt{3}}{2}\right)v \, du \, dv$$

Some details about the transformation:

- must be continuously differentiable.
- must be invertible.

Sometimes it is easier to think of the transformation going the other way.



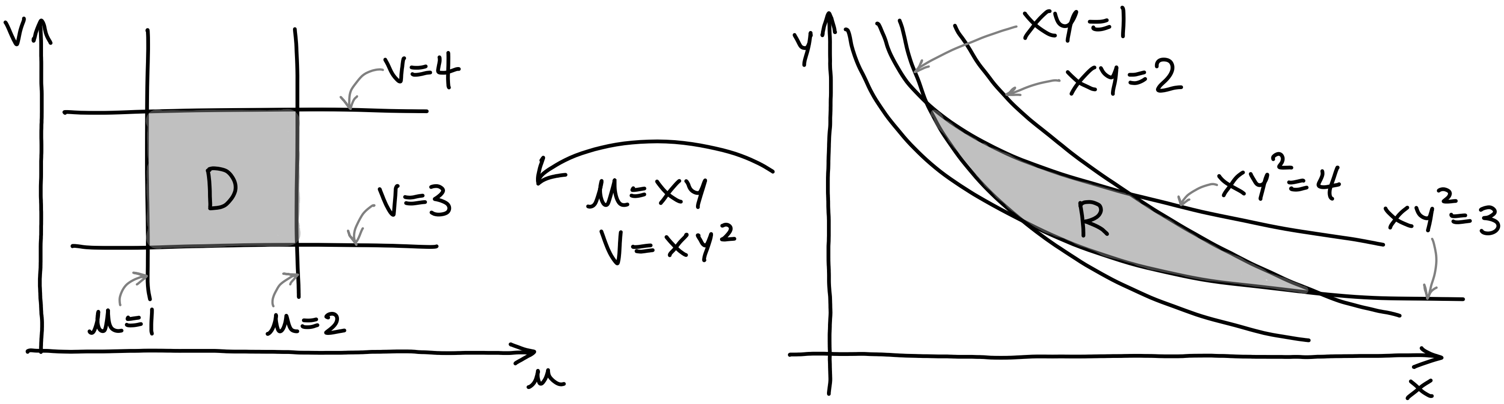
The composition does nothing, so $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 1$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \Big/ \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

So these s.f. behave like fractions – even though they aren't.

Ex:) R is bounded by $xy=1$, $xy=2$, $xy^2=3$, $xy^2=4$. in the first quadrant. Compute $\iint_R x^8 y^3 \, dA$.

Choose $u = xy$, $v = xy^2$.



Easy to compute $\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ y^2 & 2xy \end{pmatrix} = xy^2$

Then the s.f we need is $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{xy^2}$. Then

So we have

$$\iint_R x^8 y^3 \, dx\,dy = \iint_D x^8 y^3 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint_D x^8 y^3 \left(\frac{1}{xy^2} \right) du\,dv = \int_3^4 \int_1^2 x^7 y \, du\,dv$$

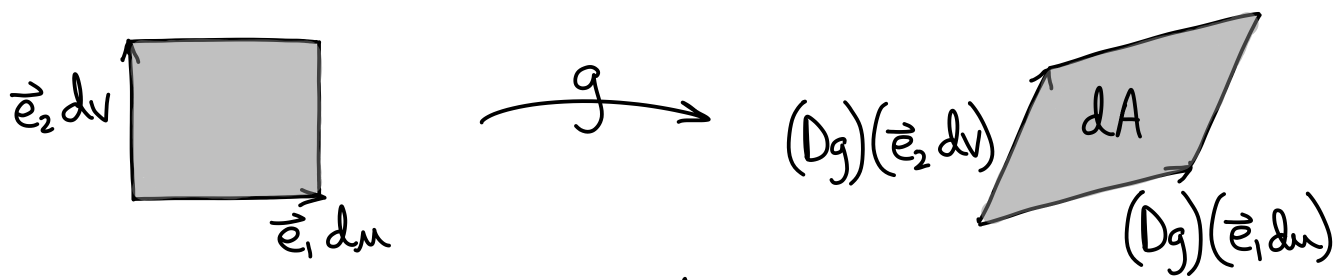
How do we convert the integrand? Try to guess the form and solve.

$$x^7 y^1 \stackrel{?}{=} (xy)^a (xy^2)^b = x^{a+b} y^{a+2b}$$

Need: $\left.\begin{aligned} a+b &= 7 \\ a+2b &= 1 \end{aligned}\right\} \Rightarrow a = 13,\ b = -6$

Then $x^7 y^1 = u^{13} v^{-6}$

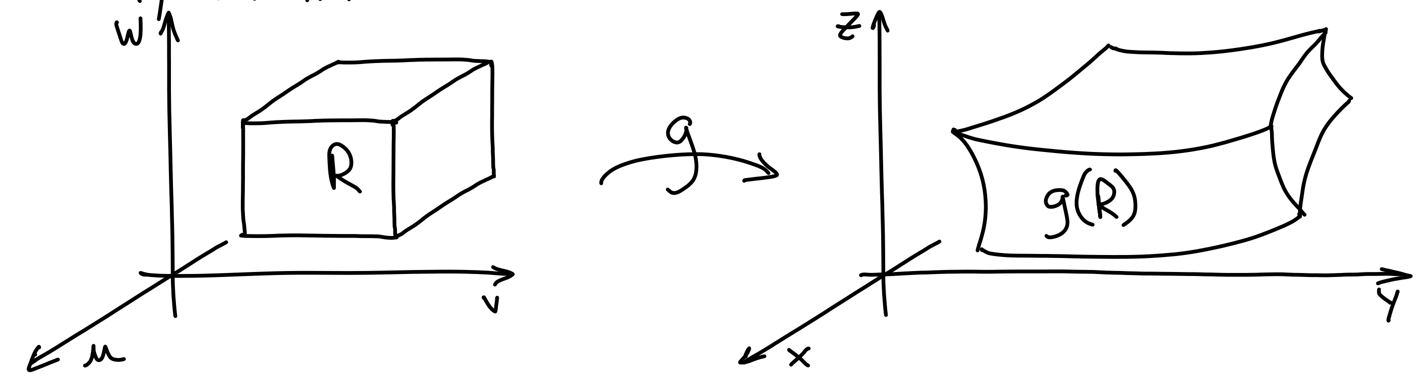
Where does the s.f. formula come from?



$$dA = \left| \det\begin{pmatrix} (Dg)(\vec{e}_1\,du) & (Dg)(\vec{e}_2\,dv) \end{pmatrix} \right| = \left| \det\begin{pmatrix} (Dg)(\vec{e}_1) & (Dg)(\vec{e}_2) \end{pmatrix} \right| du\,dv$$

$$= \left| \det(Dg) \right| du\,dv$$

Similarly in $\mathbb{R}^3$:



$$\iiint_R f(g(u,v,w))\,\left|\det Dg\right|\,du\,dv\,dw = \iiint_{g(R)} f(x,y,z)\,dV$$

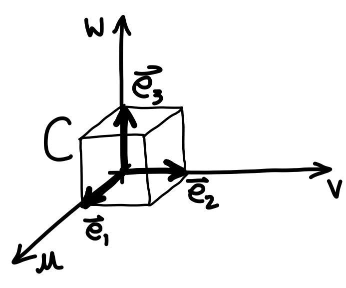
$$= \iiint_R f(g(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Thm: For $T(\vec{x}) = A\vec{x}$, $DT = A$. (Check!)

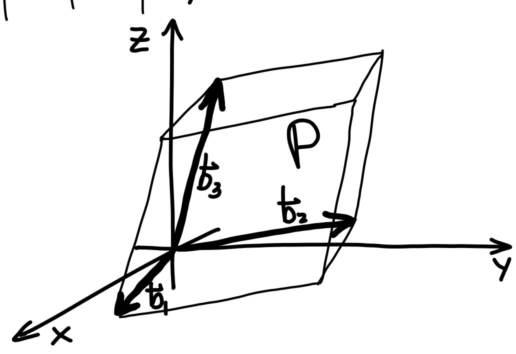
Ex: P is the parallelepiped with a corner at $\vec{0}$ and edges $\vec{b}_1 = (3,1,0)$, $\vec{b}_2 = (1,4,1)$, $\vec{b}_3 = (1,2,5)$.

Compute $\iiint_P z\,dV$.

Note that $P = T(C)$ where C is the unit cube and $T(\vec{u}) = A\vec{u} = \vec{x}$ with $A = \begin{pmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{pmatrix}$.



$\xrightarrow{T}$ $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 5 \end{pmatrix}$

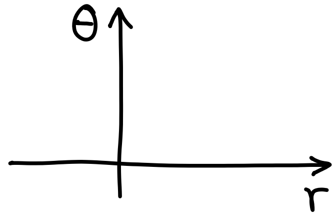


So $$\iiint_C \underbrace{z}_{=0u+1v+5w} \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{=\det A} du\,dv\,dw = \iiint_P z\,dx\,dy\,dz$$

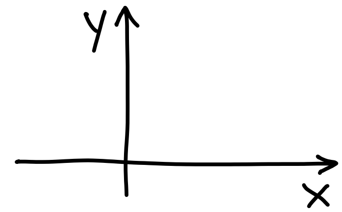
Integral "in a coordinate system"

Recall that coordinate systems are functions. We can use them as transformations for change of variables!

Polar:

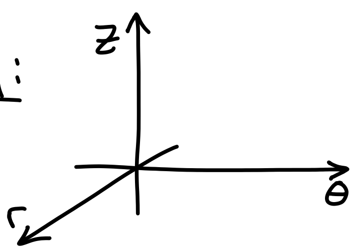


$$\xrightarrow{g_p} \quad \begin{aligned} r\cos\theta &= x \\ r\sin\theta &= y \end{aligned}$$

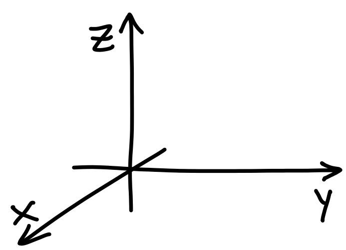


$$\frac{\partial(x,y)}{\partial(r,\theta)} = \cdots = r.$$ Choose $r \geq 0$, then $s.f. = |r| = r$.

Cylindrical:

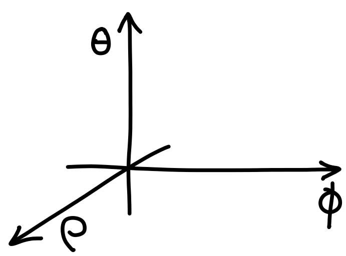


$$\xrightarrow{g_c} \quad \begin{aligned} r\cos\theta &= x \\ r\sin\theta &= y \\ z &= z \end{aligned}$$

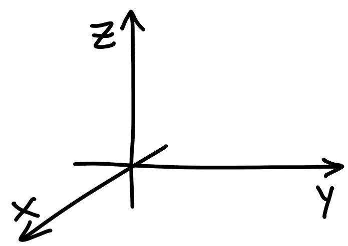


$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \cdots = r.$$ Choose $r \geq 0$, then $s.f. = |r| = r$.

Spherical:



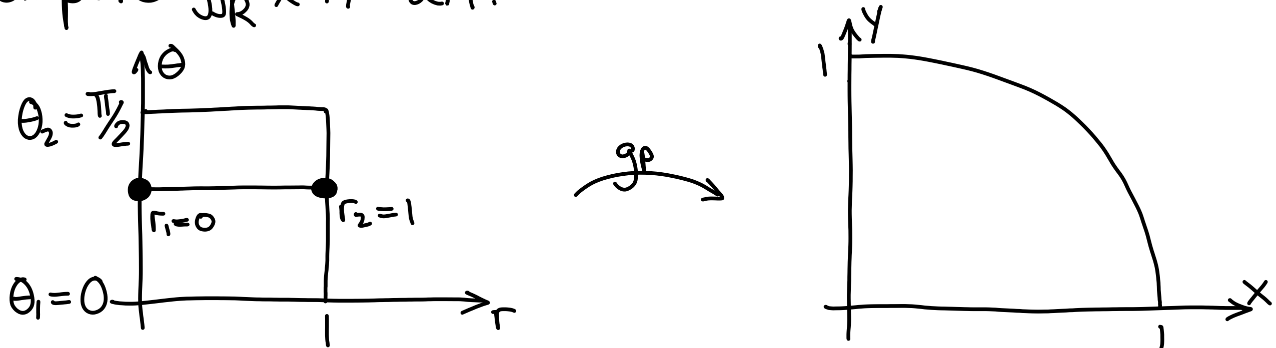
$$\xrightarrow{g_s} \quad \begin{aligned} \rho\sin\phi\cos\theta &= x \\ \rho\sin\phi\sin\theta &= y \\ \rho\cos\phi &= z \end{aligned}$$



$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \cdots = \rho^2\sin\phi.$$ Choose $\phi \in [0,\pi]$, then

$$s.f. = |\rho^2\sin\phi| = \rho^2\sin\phi$$

Ex:) R is 1st quadrant portion of the unit disk.
Compute $\iint_R x^2+y^2 \, dA$.

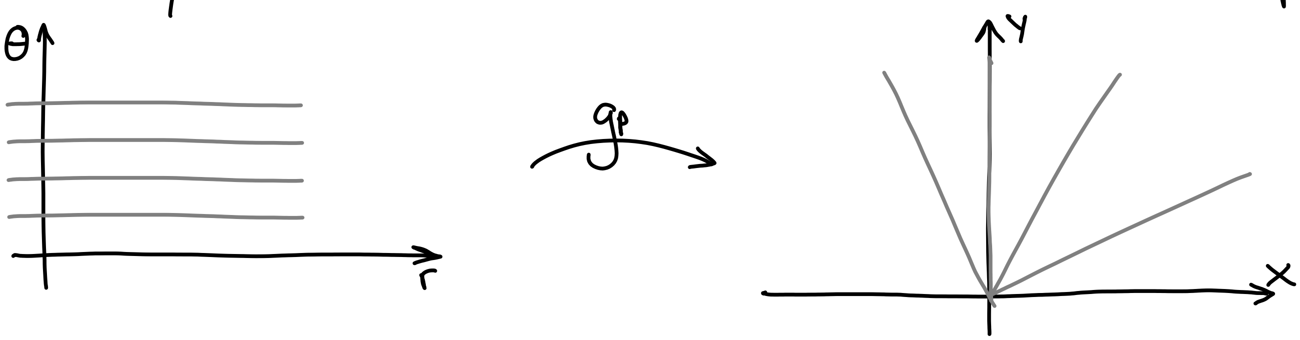


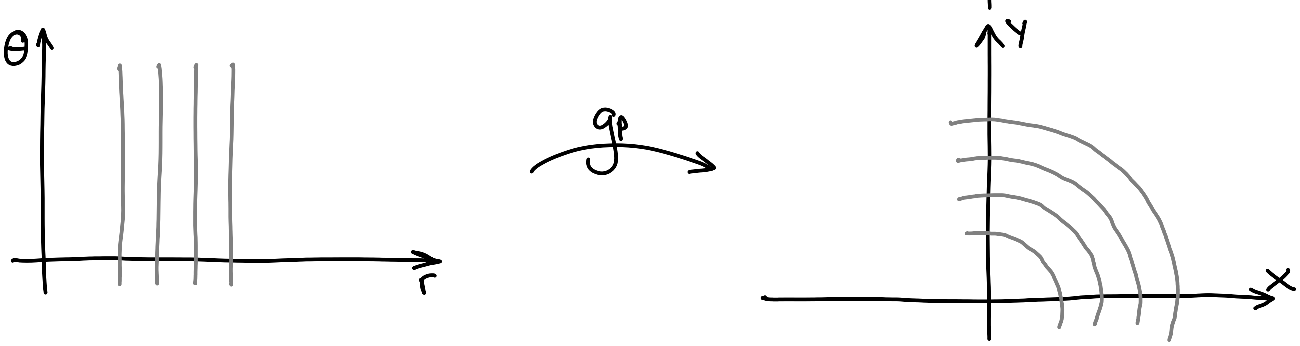
$$\int_0^{\pi/2}\int_0^1 (r^2)\,(r\,dr\,d\theta) \quad = \quad \iint_R (x^2+y^2)\,dA$$

$$\left(= \int_0^1 \int_0^{\sqrt{1-x^2}} x^2+y^2 \, dy\, dx \right.$$
$$\left. = \cdots \right)$$

$$= \int_0^{\pi/2} \left(\frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}$$

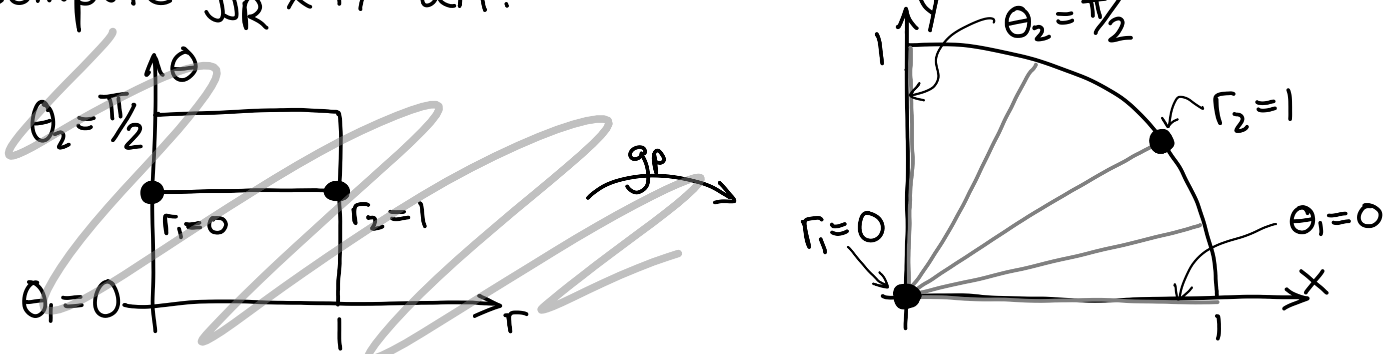
Note that you can "see" θ-slices and r-slices in the xy-plane.





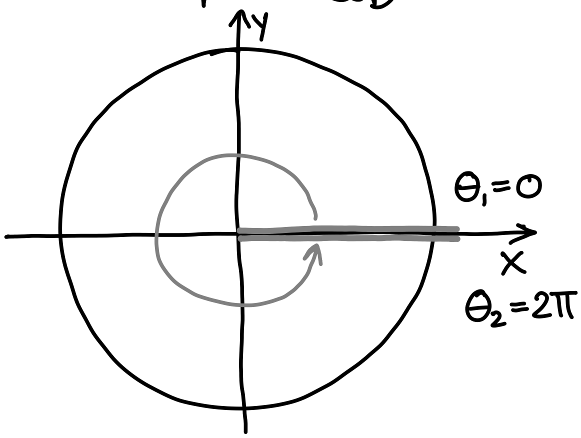
So you don't even have to draw the pull back domain to slice it!

Ex: R is 1st quadrant portion of the unit disk. Compute $\iint_R x^2+y^2 \, dA$.



$$\iint_R (x^2+y^2)\, dA = \int_0^{\pi/2}\int_0^1 (r^2)\,(r\,dr\,d\theta) = \cdots$$

Ex: Compute $\iint_D \sqrt{x^2+y^2}\, dA$ where D is the unit disk.



θ goes "all the way around".

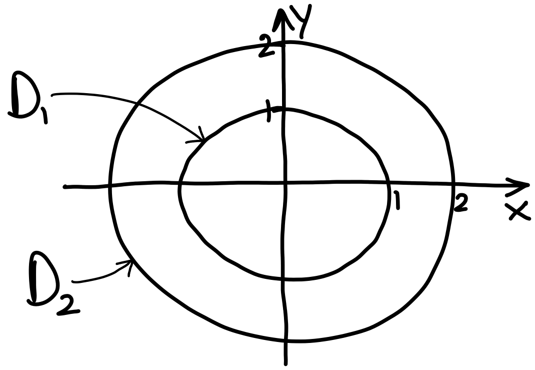
$$\int_0^{2\pi}\int_0^1 (r)(r\,dr\,d\theta) = \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{1}{3}r^3\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi}\tfrac{1}{3}\,d\theta = \frac{2\pi}{3}$$

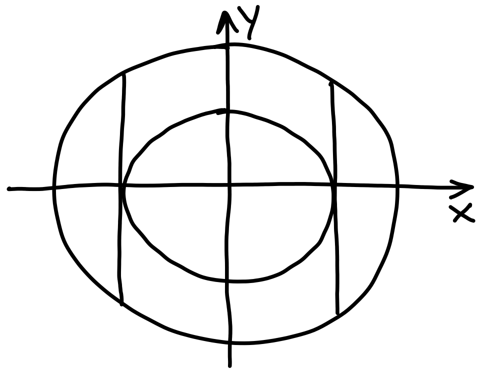
Tips

- Almost always best to use the order "$dr\,d\theta$", because r is usually dependent.
- Always write "$r\,dr\,d\theta$" all at once in an integral; forgetting the "r" is very easy/common, costs lots of points!
- Bounds come from points on curves with equations.

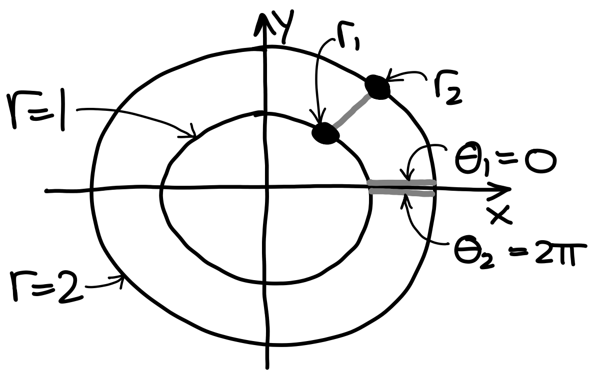
Ex:) Compute $\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA$, where R is the annulus $D_2 \setminus D_1$ below.



Tempting to say $\iint_R = \iint_{D_2} - \iint_{D_1} \ldots$ but $\frac{1}{\sqrt{x^2+y^2}}$ is undefined at $\vec{0}$!



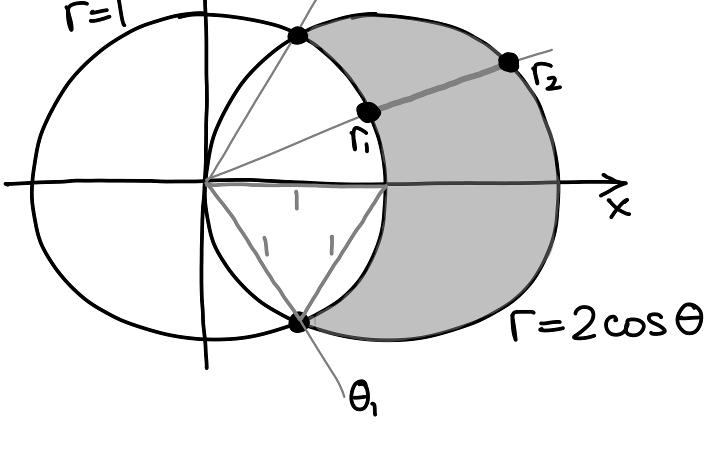
Best rectangular option requires __four__ separate (and hard!) integrals.



Easy in polar!

$$\int_0^{2\pi}\int_1^2 \left(\frac{1}{r}\right) r\,dr\,d\theta = 2\pi$$

Ex:) Compute $\iint_R x\,dA$, $R = \{(x-1)^2+y^2 \le 1\} \setminus \{x^2+y^2 \le 1\}$.



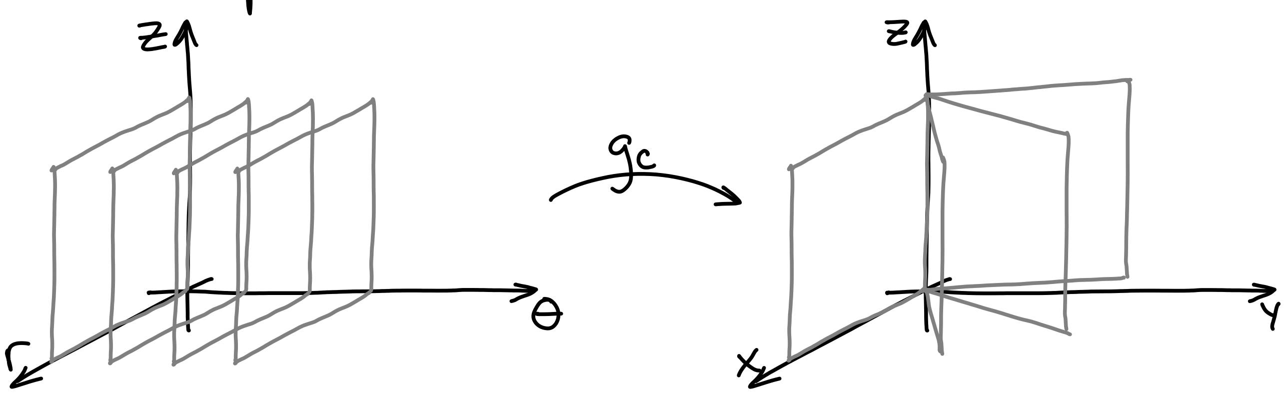
$$\int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} (r\cos\theta)(r\,dr\,d\theta)$$

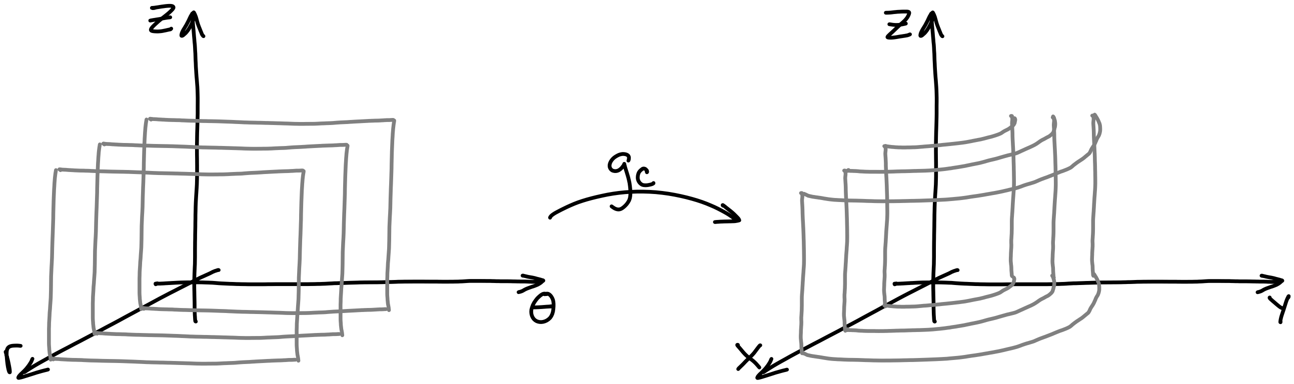
Ex:) Compute $J = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. (Relates to normal distributions)

$$
\begin{aligned}
J^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\right) dy \\
&= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = \cdots = \pi
\end{aligned}
$$

So $J = \sqrt{\pi}$.

In cylindrical too, you can "see" coordinate slices in xyz-space.





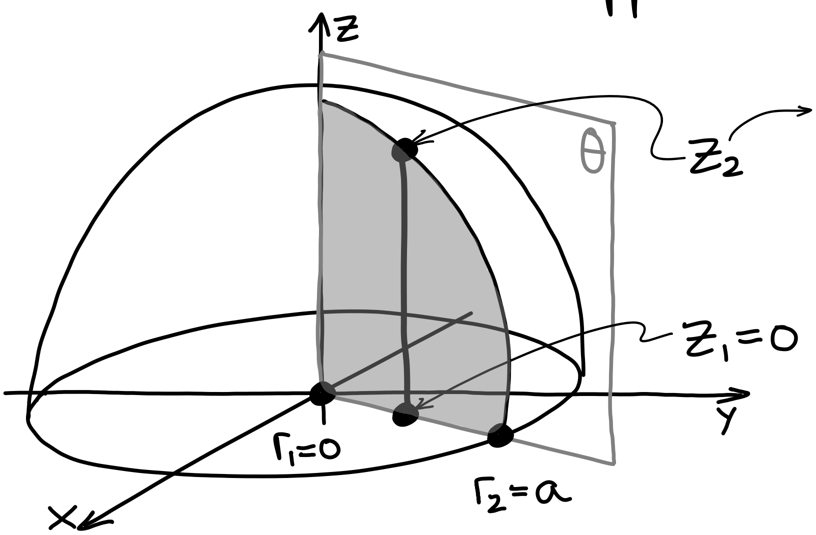
Tips for cylindrical coordinates

- Almost always best to use "$dr\,d\theta\,dz$" or "$dz\,dr\,d\theta$" because r is usually dependent on θ.

- Always write "$r\,dr\,d\theta\,dz$" all at once in an integral; forgetting the "r" is very easy/common, costs lots of points!

- Bounds come from points on surfaces with equations.

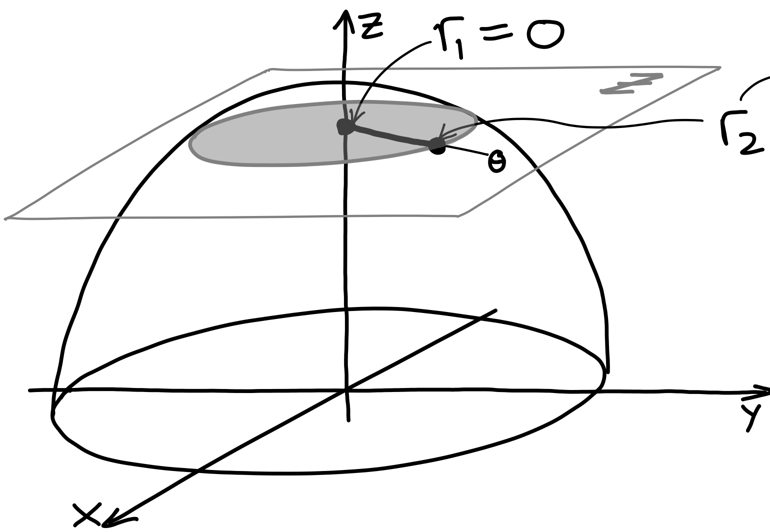
Ex:) Let R be the upper half ball of radius a.



on sphere $r^2+z^2=a^2$

$$\Rightarrow z_2=\sqrt{a^2-r^2}$$

$$\iiint_R f\,dV = \int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} f\,r\,dz\,dr\,d\theta$$

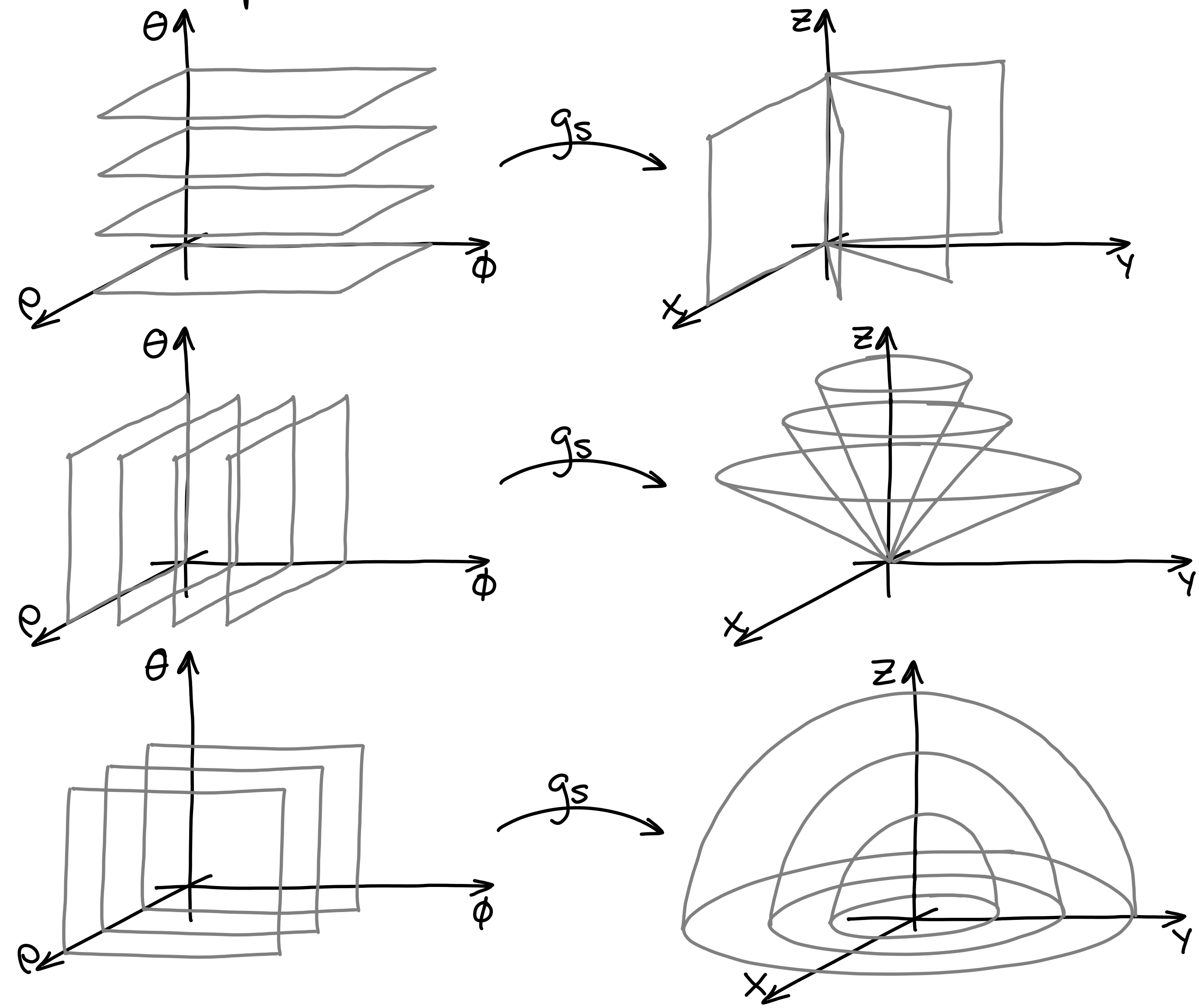


on sphere $r^2+z^2=a^2$

$$\Rightarrow r_2=\sqrt{a^2-z^2}$$

$$\iiint_R f\,dV = \int_0^{a}\int_0^{2\pi}\int_0^{\sqrt{a^2-z^2}} f\,r\,dr\,d\theta\,dz$$

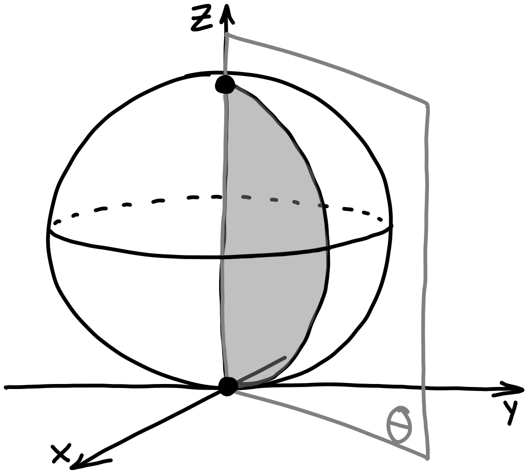
In spherical too, you can "see" coordinate slices in XYZ-space.



Tips for spherical coordinates

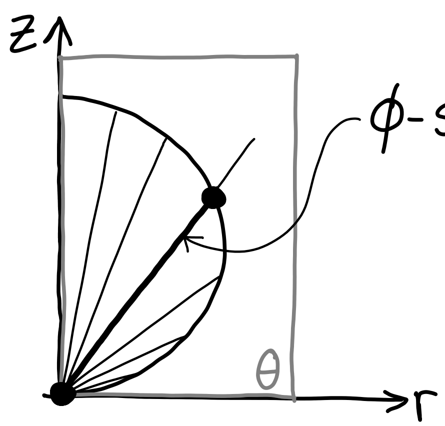
- Almost always best to use "$d\rho\, d\phi\, d\theta$", because ρ is usually dependent on ϕ, θ, and given shapes of slices.
- Always write "$\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$" all at once in an integral; forgetting the "$\rho^2 \sin\phi$" is very easy/common, costs lots of points!
- Bounds come from points on surfaces with equations.

Ex:) Let R be the ball, radius a, center $(0,0,a)$.



$\theta_1 = 0$

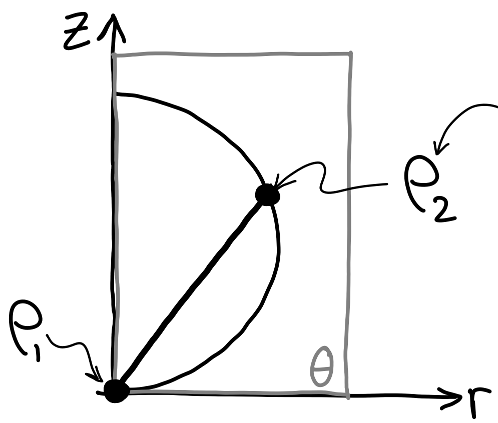
$\theta_2 = 2\pi$



ϕ-slice of this θ-slice

$\phi_1 = 0$

$\phi_2 = \pi/2$



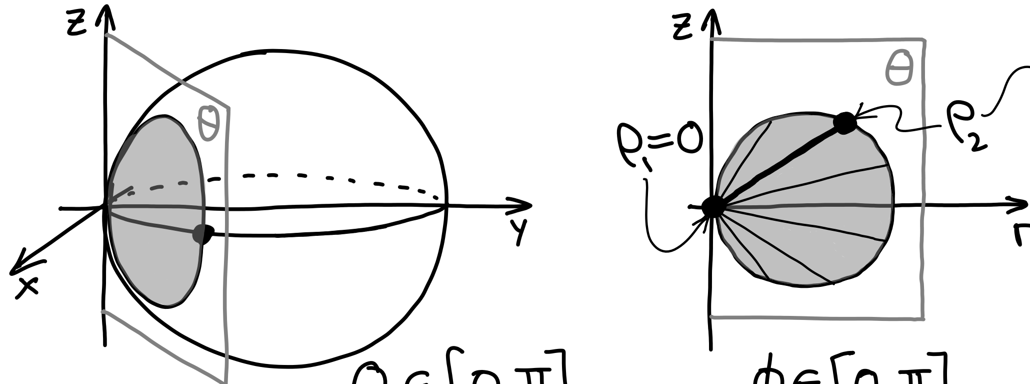
on sphere $\rho = 2a\cos\phi$. So

$$\iiint_R f \, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi} f \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

A good picture is key!

- Make it big, clear.
- Draw more than one, revise/redo, as needed.
- Learn technique, + practice!

Ex:) Let R be the ball, radius a, center $(0,a,0)$.



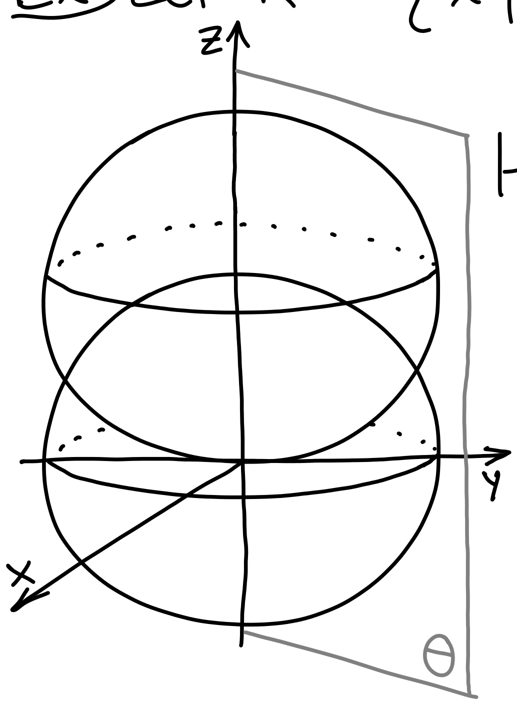
$\rho_1 = 0$

ρ_2 on sphere $\rho = 2a\sin\phi\sin\theta$ (Derive!)

$\theta \in [0,\pi]$ $\qquad$ $\phi \in [0,\pi]$

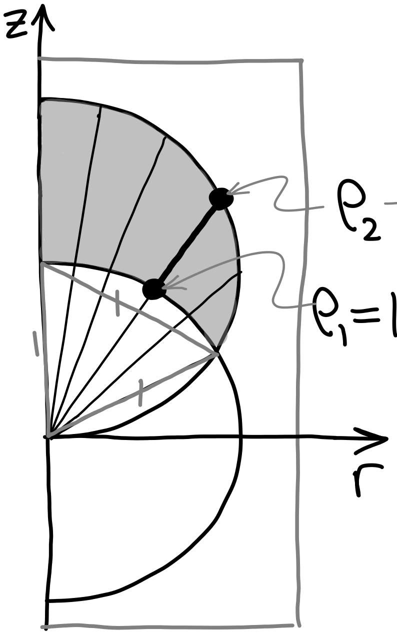
So $\iiint_R f\, dV = \int_0^{\pi}\int_0^{\pi}\int_0^{2a\sin\phi\sin\theta} f\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

Ex:) Let $R = \{\vec{x} \mid x^2+y^2+z^2 \geq 1,\ x^2+y^2+(z-1)^2 \leq 1\}$



Hard to draw, but:

- Clear that $\theta \in [0, 2\pi]$,
- A cross section of the intersection is the intersection of the cross sections.



$\phi \in [0, \pi/3]$

ρ_2 on sphere $\rho = 2\cos\phi$. So

$$\iiint_R f\, dV =$$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_1^{2\cos\phi} f\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

Symmetry theorems

In single variable:

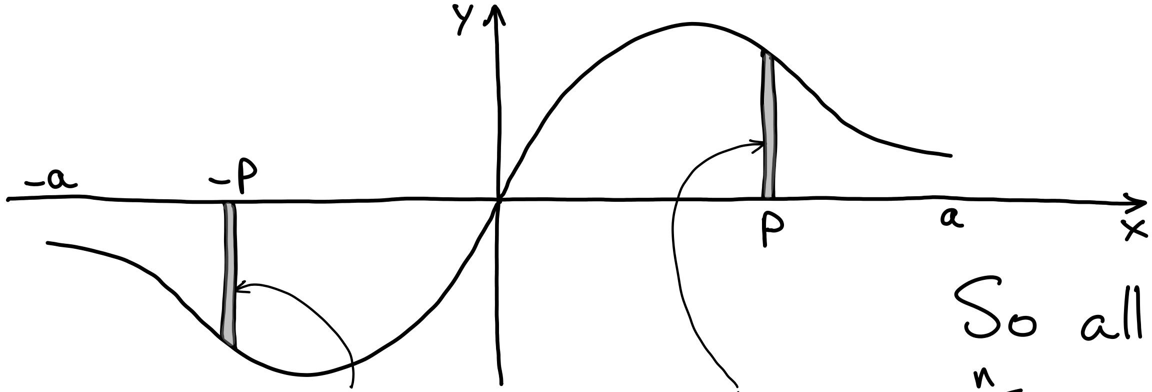
① f is odd iff $f(-x) = -f(x)$ (for all x).

② interval I is symmetric iff $(x \in I \Leftrightarrow -x \in I)$.

Thm:) If ① f is odd
② I is symmetric

then $\int_I f(x)\, dx = 0$

Can see this from a Riemann sum point of view:



$$f(-p)\,\Delta x = -f(p)\,\Delta x$$

So all terms of $\sum_{i=1}^{n} f(x_i)\Delta x$ cancel!

We can also prove it with a substitution $x = g(u) = -u$ thought of as a change of variables:

$$\int_I f(x)\,dx = \int_{g(I)} f(x)\,dx \quad \text{(symmetry of } I\text{)}$$

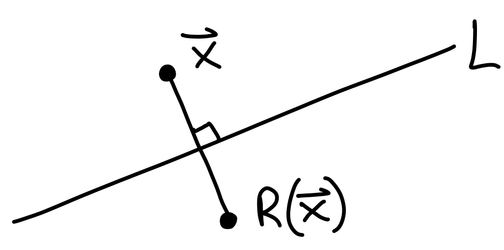
$$= \int_I f(g(u))\,(1)\,du \quad \text{(pull back)}$$

$$= \int_I f(-u)\,du = -\int_I f(u)\,du \quad \text{(oddness of } f\text{)}$$

$$\Rightarrow \int_I f\,dx = 0$$

<u>In $\mathbb{R}^2$:</u>

<u>Def:</u> The reflection R over the line L is



Def: f has odd symmetry over L iff
$$f(R(\vec{x})) = -f(\vec{x}) \qquad \text{for all } \vec{x}$$

Def: D is symmetric over L iff
$$(\vec{x} \in D) \iff (R(\vec{x}) \in D) \qquad \text{for all } \vec{x}$$

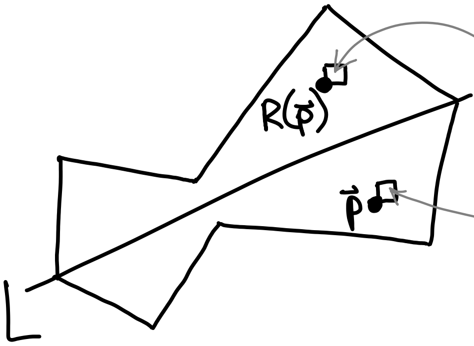
Thm: If ① f is odd over L
② D is symmetric over L

then $\iint_D f \, dA = 0$

NB – Both conditions must hold!
Must use the same line for each!

Pf: $\iint_D f(\vec{x})\,dx\,dy = \iint_{R(D)} f(\vec{x})\,dx\,dy$
$= \iint_D f(R(\vec{u}))\,(1)\,du\,dv$
$= -\iint_D f(\vec{u})\,du\,dv \qquad \Longrightarrow \iint_D f(\vec{x})\,dx\,dy = 0$

Riemann sum point of view:



$$f(R(\vec{p}))\,\Delta A = -f(\vec{p})\,\Delta A$$

So all terms of
$$\sum_{i,j=1}^{n} f(\vec{x}_{i,j})\,\Delta A \text{ cancel!}$$

Ex:) D is the region bounded by $y=x^2-x^4$, $y=-1$.
Compute $\iint_D x^3e^{x^2+1}-y\sin x \, dA$.

Note that

① D is symmetric over the y-axis $(R(x,y)=(-x,y))$.
(Both equations are unaffected by the reflection.
E.g.: $y=(-x)^2-(-x)^4 \iff y=x^2-x^4$)

② f is odd over the y-axis.

$$f(R(x,y)) = f(-x,y) = (-x)^3e^{(-x)^2+1} - y\sin(-x)$$
$$= -x^3e^{x^2+1} + y\sin(x) = -f(x,y)$$

Then $\iint_D x^3e^{x^2+1}-y\sin x \, dA = 0$ by symmetry.

A complete symmetry argument <u>must</u> include:

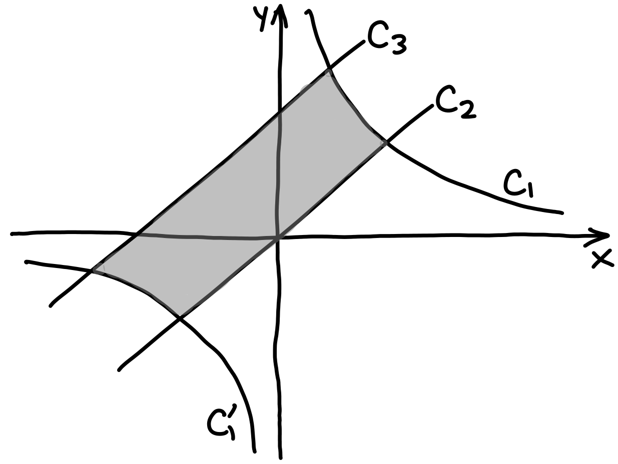
① the line of symmetry L
② the reflection $R(x,y)$
③ (correct) observation that f is odd
④ demonstration that f is odd
⑤ (correct) observation that D is symmetric.

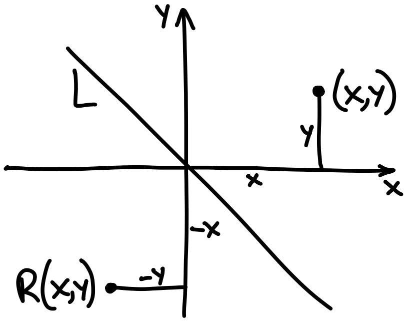
Be careful using symmetry! If you are mistaken and the argument doesn't work, none of it is actual progress!

Ex:) Compute $\int_0^1\int_0^1 x^3-y^3 \, dx\,dy$.

$\left.\begin{array}{l} D=[0,1]\times[0,1] \text{ is symmetric} \\ f(x,y)=x^3-y^3 \text{ has odd symmetry} \\ \Rightarrow \iint=0 \text{ by symmetry.} \end{array}\right\}$ Only shows 2/5 of the required parts!

Ex:) D is bounded by $xy=1$, $y=x$, $y=x+1$.
Compute $\iint_D (x+y)^7 (x-y)^3 \, dA$.





Consider $L=\{x+y=0\}$, $R(x,y)=(-y,-x)$.

C_1, C_1' are reflections of each other; C_2, C_3 are reflections of themselves. So D is symmetric over L.

And f is odd because

$$f(R(x,y)) = f(-y,-x) = ((-y)+(-x))^7((-y)-(-x))^3$$
$$= -(x+y)^7 (x-y)^3 = -f(x,y)$$

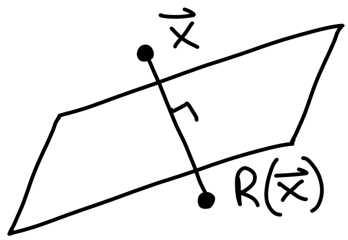
So $\iint = 0$ by symmetry.

Ex:) Triangle T has corners $(0,2)$, $(-1,0)$, $(1,0)$.
Compute $\iint_T y^3 \, dA$.

T is symmetric over the y-axis only.
$f(x,y)=y^3$ is odd over the x-axis only.
$\Big\}$ No symmetry argument works!!

In $\mathbb{R}^3$:

Def: The reflection R over the plane P is:



Def: f has odd symmetry through P iff:

$$f(R(\vec{x})) = -f(\vec{x}) \qquad \text{for all } \vec{x}$$

Def: D is symmetric through P iff:

$$(\vec{x} \in D) \iff (R(\vec{x}) \in D) \qquad \text{for all } \vec{x}$$

Thm: If
1. f is odd through P
2. D is symmetric through P

then $\iiint_D f\, dV = 0$

NB– Both conditions must hold!
Must use the same plane for each!

Ex: B is the unit ball, compute $\iiint_B x^2y^2z^3\, dV$.

P is the xy-plane, $R(x,y,z) = (x,y,-z)$, B is symmetric.

f is odd because

$$f(R(x,y,z)) = f(x,y,-z) = (x)^2(y)^2(-z)^3 = -x^2y^2z^3 = -f(x,y,z)$$

So $\iiint = 0$ by symmetry.

5.6 - Applications of Integration

Average value of a function

Averaging a collection of numbers with repetitions can be thought of as a "weighted average".

$$\frac{4+4+7+8+8+8}{6} = \frac{4+4+7+8+8+8}{1+1+1+1+1+1} = \frac{4\cdot 2+7\cdot 1+8\cdot 3}{2+1+3}$$

In general, the weighted average of $v_1,\dots,v_n$ with weights $w_1,\dots,w_n$ (resp.) is

$$\bar{v} = \frac{\sum v_i w_i}{\sum w_i}$$

Ex:) Exams 1,2,3,4 in a class have weights 20, 20, 20, 40. Bob's grades are 2.06, 3.25, 2.71, 3.40. What is his course average?

$$\bar{g} = \frac{\sum g_i w_i}{\sum w_i} = \frac{(2.06)(20)+(3.25)(20)+(2.71)(20)+(3.40)(40)}{(20)+(20)+(20)+(40)} = 2.964$$

For $f:\mathbb{R}^n \to \mathbb{R}^1$, we use size in the domain as the weights.

Ex:) The average value of $f:\mathbb{R}^1 \to \mathbb{R}^1$ on $[a,b]$ is

$$\bar{f} = \frac{\int_a^b f(x)\,dx}{\int_a^b dx} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

So the average value of $\sin x$ on $[0,\pi]$ is

$$\frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}$$

Ex:) The average value of $f: \mathbb{R}^2 \to \mathbb{R}^1$ on $D \subset \mathbb{R}^2$ is

$$\bar{f} = \frac{\iint_D f(\vec{x})\, dA}{\iint_D dA} = \frac{1}{\text{area}} \iint_D f(\vec{x})\, dA$$

So the average value of x^2 on the unit disk is

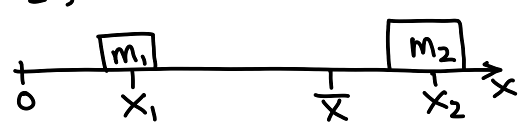
$$\frac{1}{\pi}\iint_D x^2\, dA = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 (r\cos\theta)^2\, r\, dr\, d\theta = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\, dr\, d\theta = \frac{1}{4}$$

Center of mass

The center of mass is the weighted average of position.

Ex:) Mass m_1 at x_1, m_2 at x_2, then

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$



Physical intuition: $\bar{x}$ is the balance point; total torque $= 0$.

In general, $$\bar{\vec{x}} = \frac{\iint_D \vec{x}\, dm}{\iint_D dm} = \frac{1}{m}\iint_D \vec{x}\, dm$$

And you can compute the coordinates separately.

$$\bar{\vec{x}} = \left(\frac{1}{m}\iint_D x_1\, dm, \ldots, \frac{1}{m}\iint_D x_n\, dm\right)$$

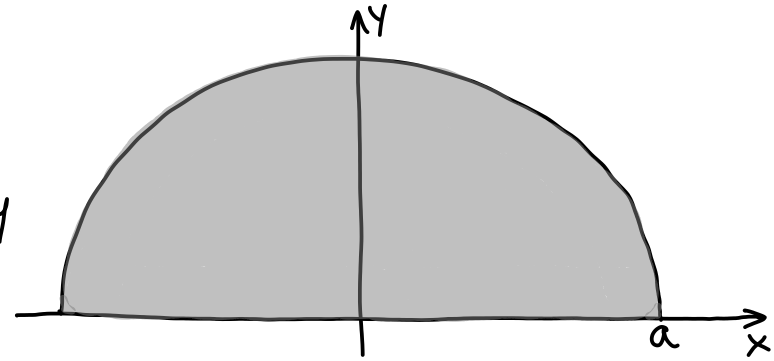
Ex:) D is the upper half disk, radius $= a$, $\delta = 1$.

① $\bar{x} = \frac{1}{A}\iint_D x\, dA$ $= 0$ by symmetry (details!)

② $\overline{y} = \frac{1}{A}\iint_D y\,dA$

$$= \frac{1}{\frac{1}{2}\pi a^2}\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} y\,dx\,dy$$

$$= \frac{1}{\frac{1}{2}\pi a^2}\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} y\,dy\,dx$$



(Which of these is easier to evaluate?) $\left(\cdots = \frac{4a}{3\pi}\right)$

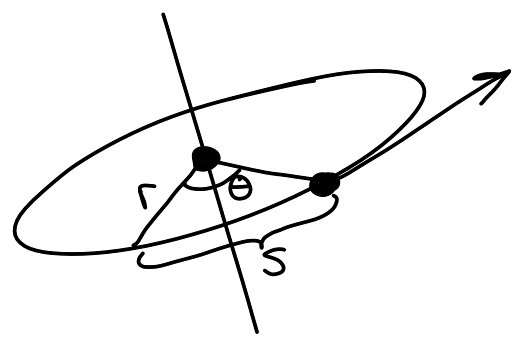
Moment of inertia

Particle rotating around an axis...

$$s = r\theta$$
$$s' = r\theta'$$
$$v = r\omega$$

speed

angular speed



What is the kinetic energy of a solid mass rotating around an axis?

Can't use $KE = \frac{1}{2}mv^2$ on the whole, because speed is not constant.

But on pieces it is!

$$KE = \iiint d(KE) = \iiint \frac{1}{2}v^2\,dm = \iiint \frac{1}{2}(r\omega)^2\,dm$$

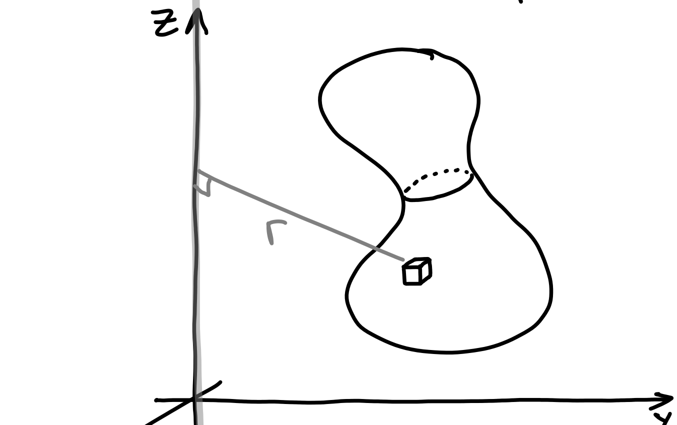
$$= \frac{1}{2}\left(\iiint r^2\,dm\right)\omega^2 = \frac{1}{2} I \omega^2$$

moment of inertia

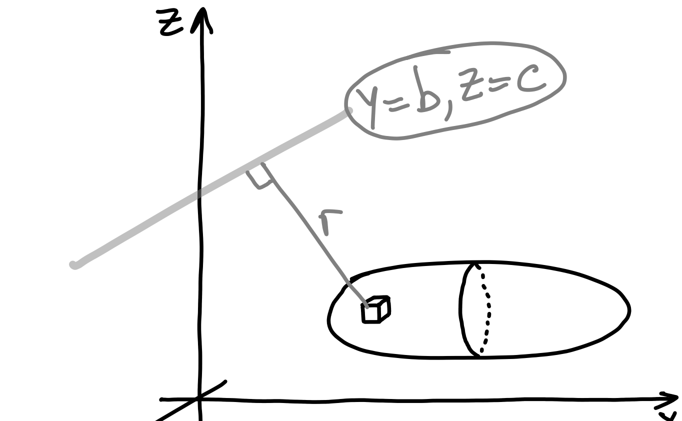
angular analogue of mass

angular analogue of speed

What is r? Depends on the axis of rotation.

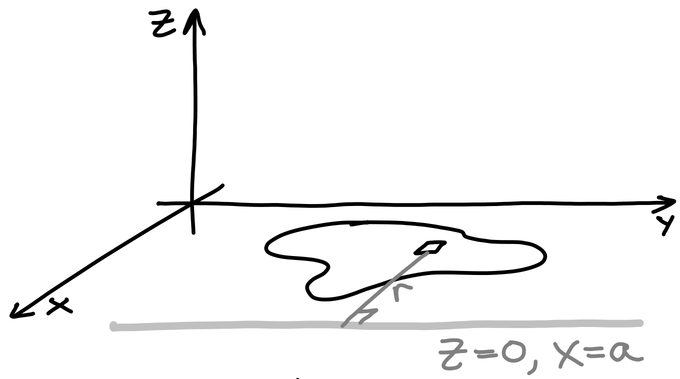


$r=\sqrt{x^2+y^2}$



$r=\sqrt{(y-b)^2+(z-c)^2}$

$r=|x|$



$r=|x-a|$

Ex: $D=\{(x,y) \mid 0\le x\le \pi,\ 0\le y\le \sin x\}$, $\delta = 2+\cos x$.
Find the moment of inertia around the x-axis.

$$I=\iint_D r^2\,dm=\iint_D r^2\,\delta\,dA$$

$$=\int_0^{\pi}\int_0^{\sin x} y^2\,(2+\cos x)\,dy\,dx$$

$$=\frac{1}{3}\int_0^{\pi}(\sin^3 x)(2+\cos x)\,dx$$

$$=\cdots$$