4.1 – Differentials and Taylor's Theorem

Recall from single variable calculus, the Taylor polynomial of degree k for $f \in C^k$ at a is:

$$p_k(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{k!} f^{[k]}(a)(x-a)^k$$

This is the unique polynomial of degree k that "mimics" f:

$$p_k(a) = f(a), \quad p_k'(a) = f'(a), \quad \cdots, \quad p_k^{[k]}(a) = f^{[k]}(a)$$

For multivariable functions, we similarly define the Taylor polynomial of degree k as the unique polynomial of degree k that "mimics" f up to kth order (partial) derivatives.

$$p_k(\vec{a}) = f(\vec{a}), \quad \frac{\partial}{\partial x_i} p_k(\vec{a}) = \frac{\partial}{\partial x_i} f(\vec{a}), \quad \cdots, \quad \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} p_k(\vec{a}) = \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} f(\vec{a})$$

You can check directly that

$$p_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot Hf(\vec{a})(\vec{x}-\vec{a})$$

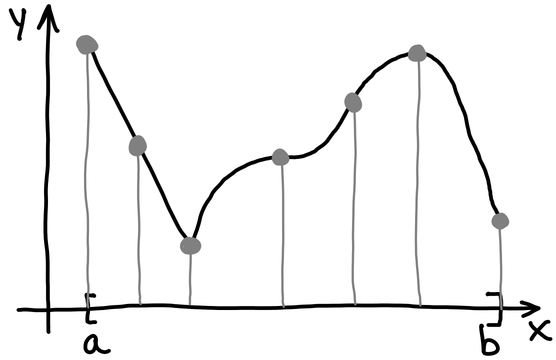
where

$$Hf(\vec{a}) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_k} \\ f_{x_2x_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ f_{x_kx_1} & & \cdots & f_{x_kx_k} \end{pmatrix}$$

(Notice, $p_1(\vec{x})$ is just the linear approximation!)

4.2 – Extrema of Functions

Recall from single variable calculus:



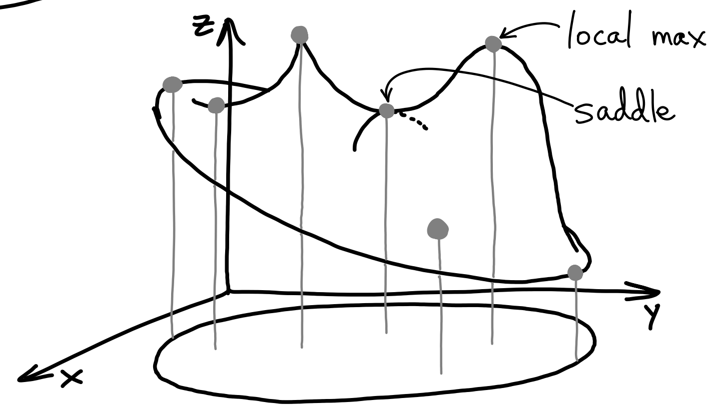
① x in interior of $[a,b]$
② f diff'bl at x
③ $f'(x) \neq 0$

$\Rightarrow$ f is increasing (one way or other) $\Longrightarrow$ x is <u>not</u> a maximizer of f on $[a,b]$.

So if ($-$① x on boundary of $[a,b]$
or $-$② f not diff'bl at x
or $-$③ $f'(x) = 0$)... then we cannot eliminate x as a possible maximizer.

critical points

Similarly in multivariable:



① $\vec{x}$ in interior of D
② f diff'bl at $\vec{x}$
③ $\nabla f(\vec{x}) \neq \vec{0}$

$\Rightarrow$ f is increasing (one way or other) $\Longrightarrow$ $\vec{x}$ is <u>not</u> a maximizer of f on D.

So if ($-$① $\vec{x}$ on boundary of D
or $-$② f not diff'bl at $\vec{x}$
or $-$③ $\nabla f(\vec{x}) = \vec{0}$)... then we cannot eliminate $\vec{x}$ as a possible maximizer.

critical points

(image search: "monkey saddle")

Ex:) Find critical points of $f(x,y) = x^3 - 3x + y^2$

(-1) Defined on $\mathbb{R}^2$, every point is interior.

(-2) Polynomial, so diff'bl everywhere.

(-3) $\nabla f = \begin{pmatrix} 3x^2-3 \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x = \pm 1 \\ y = 0 \end{matrix}$ So c.p.'s are $(-1,0), (1,0)$.

Sometimes the algebra requires considering cases.

Ex:) How do we solve

$$\begin{pmatrix} 3(x^2-y^2) \\ 6xy+6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ? \Rightarrow x^2 = y^2 \Rightarrow \begin{matrix} (1)\ x = y \\ \text{or} \\ (2)\ x = -y \end{matrix}$$

(1) $x = y \Rightarrow 6xy + 6y = 6y^2 + 6y = 0 \Rightarrow y = 0$ or -1.

c.p.'s: $(0,0), (-1,-1)$.

(2) $x = -y \Rightarrow 6xy + 6y = -6y^2 + 6y = 0 \Rightarrow y = 0$ or 1.

c.p.'s: $(0,0), (-1,1)$.

2nd derivative test

In single variable, the behavior of f at a critical point is dominated by the 2nd order term of the Taylor series.

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Similarly for multivariable functions:

$$p_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})\cdot Hf(\vec{a})(\vec{x}-\vec{a})$$

$$Hf(\vec{a}): \begin{cases} \text{pos. def.} \Rightarrow & \text{local minimum} \\ \text{neg. def.} \Rightarrow & \text{local maximum} \\ \text{indefinite} \Rightarrow & \text{saddle point} \end{cases}$$

For $f:(D\subset\mathbb{R}^2)\to\mathbb{R}$, using some linear algebra (Hf symm $\Rightarrow$ diag'bl, $\det(Hf)=\lambda_1\lambda_2$, $\vec{z}\cdot(Hf)\vec{z} = \lambda_1 z_1^2 + \lambda_2 z_2^2$), we can boil this down to:

2nd deriv. test: (interior of $D\subset\mathbb{R}^2$, $Df=0$)

$$\det(Hf)>0 \Rightarrow \begin{cases} f_{xx}>0 \Rightarrow \text{local min.} \\ f_{xx}<0 \Rightarrow \text{local max.} \end{cases}$$

$$\det(Hf)<0 \Rightarrow \text{saddle point}$$

$$\det(Hf)=0 \Rightarrow \text{test fails}$$

Ex:) We previously saw that $f:\mathbb{R}^2\to\mathbb{R}^1$ defined by $f(x,y) = x^3-3x+y^2$ has critical points at $\vec{a}=(-1,0)$, $\vec{b}=(1,0)$.

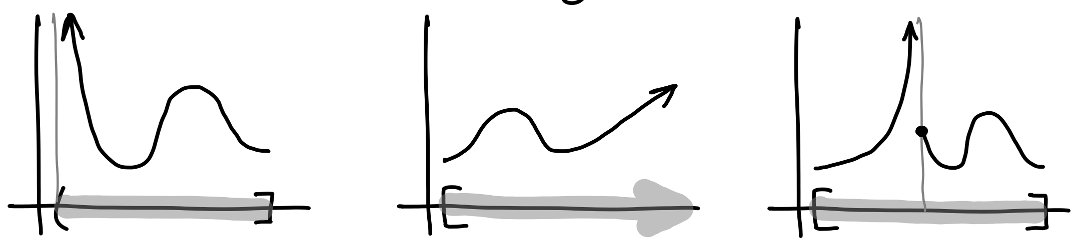
Note that $Hf = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \det(Hf) = 12x$. Then

$\vec{a}=(-1,0)$: $\det = -12 < 0 \Rightarrow$ saddle

$\vec{b}=(1,0)$: $\det = 12 > 0$, $f_{xx} = 6 > 0 \Rightarrow$ local min.

Global extrema

Is the greatest local max also a global max?
First we need to know a global max exists!



Thm: For $f:(D\subset\mathbb{R}^n)\to\mathbb{R}$, if

1. D contains its boundary ("closed")
2. D is a subset of a ball ("bounded")

} ("compact")

3. f is continuous

then there is a global max (and min) of f on D.

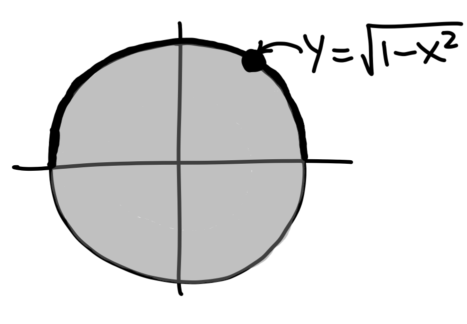
We also need to be able to check the boundary somehow... We will find better methods for this in 4.3.

Ex: Maximize $f(x,y)=x^2+x+y^2-y$ on $D=\{x^2+y^2\le 1\}$.

f diff'bl everywhere.

$$\nabla f=\vec{0}\Rightarrow\begin{pmatrix}2x+1\\2y-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1/2\\1/2\end{pmatrix},\ \in D\Rightarrow\text{critical point.}$$

On ∂D $x^2+y^2=1$ so $f(x,y)=1+x-y$.



<u>top edge:</u> $y=\sqrt{1-x^2}$, then for $x\in[-1,1]$ we have

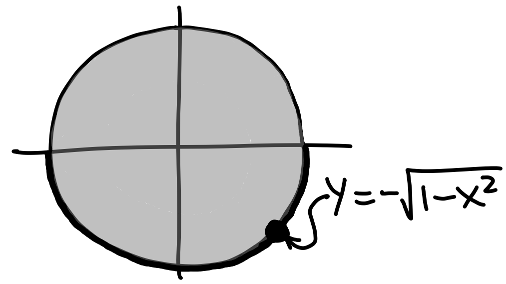
$$f(x,\sqrt{1-x^2})=1+x-\sqrt{1-x^2}$$

$$df/dx = 1 - \frac{1}{2}\frac{-2x}{\sqrt{1-x^2}} = 1 + \frac{x}{\sqrt{1-x^2}}$$

$$df/dx = 0 \Rightarrow x = -\sqrt{1-x^2} \Rightarrow x^2 = 1-x^2 \Rightarrow x = \pm\sqrt{1/2}$$

And of course we also check endpoints, $x = \pm 1$.

<u>bottom edge</u>: $y = -\sqrt{1-x^2}$, then for $x \in [-1,1]$ we have



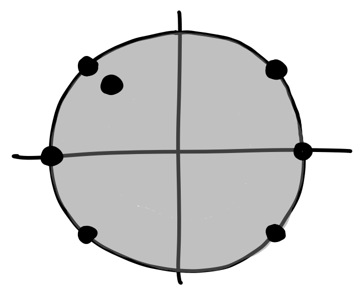
$$f(x, -\sqrt{1-x^2}) = 1 + x + \sqrt{1-x^2}$$

$$df/dx = 1 + \frac{1}{2}\frac{-2x}{\sqrt{1-x^2}} = 1 - \frac{x}{\sqrt{1-x^2}}$$

$$df/dx = 0 \Rightarrow x = \sqrt{1-x^2} \Rightarrow x^2 = 1-x^2 \Rightarrow x = \pm\sqrt{1/2}$$

And of course we also check endpoints, $x = \pm 1$.

All together we have 7 points not eliminated.



D is closed + bounded, and f is continuous, so the absolute max <u>does</u> exist.

So it is whichever of the above gives greatest value of f.

4.3 – Lagrange Multipliers

Sometimes we want to optimize a function, but <u>with a constraint</u>.

<u>Ex:</u>) Find the point on $x^2+y^2-3xy = 4$ that is closest to the origin.

fn to optimize (minimize) : $f(x,y) = x^2+y^2$ ("objective function")

fn whose <u>level set</u> is the constraint : $g(x,y) = x^2+y^2-3xy$ ("constraint function")

<u>Ex:</u>) Find the biggest (volume) box with a given area.

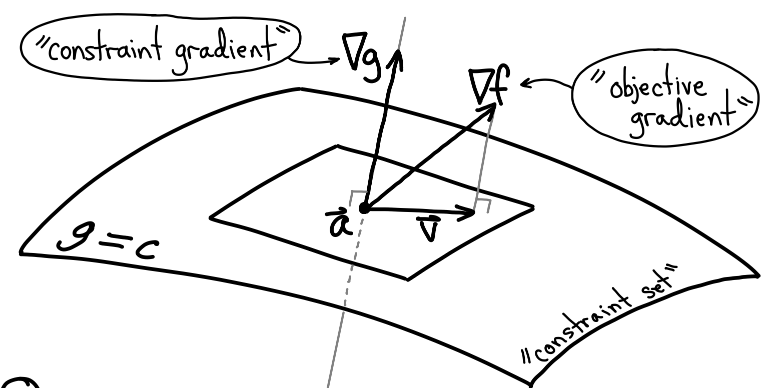
<u>Ex:</u>) How to allocate a limited budget toward labor, tools, materials to maximize productivity?

<u>Key tool</u>: If

① f, g diff'bl

② $\nabla g \neq \vec{0}$

③ $\nabla f \neq \lambda \nabla g$



then $D_{\vec{v}} f(\vec{a}) = \nabla f \cdot \vec{v} > 0$

So f is increasing, thus $\vec{a}$ is <u>not</u> a maximizer!

If (ⓘ f, g not diff'bl **or** ② $\nabla g = \vec{0}$ **or** ③ $\nabla f = \lambda \nabla g$) … then we cannot eliminate $\vec{x}$ as a possible maximizer.

critical points

Ex:) Find the highest point on the surface S,
$x^2+y^2+3z^2+xz-2yz=4$.

$f(x,y,z)=z$, $g(x,y,z)=x^2+y^2+3z^2+xz-2yz$, both diff'bl.

$$\nabla f=\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \nabla g=\begin{pmatrix}2x+z\\2y-2z\\6z+x-2y\end{pmatrix}=\begin{pmatrix}2&0&1\\0&2&-2\\1&-2&6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\vec{0}$$

iff $\vec{x}=\vec{0}$ b/c det $\neq 0$

not on S!

$$\nabla f=\lambda\nabla g \Rightarrow \begin{cases}0=\lambda(2x+z) & \text{(a)}\\ 0=\lambda(2y-2z) & \text{(b)}\\ 1=\lambda(6z+x-2y) & \text{(c)}\end{cases}$$

$$\text{Constraint} \Rightarrow \quad 4=x^2+y^2+3z^2+xz-2yz \qquad \text{(d)}$$

(c) $\Rightarrow \lambda\neq 0$

(a),(b) $\Rightarrow$ $\begin{matrix}2x+z=0\\2y-2z=0\end{matrix} \Rightarrow \begin{matrix}x=-\frac{1}{2}z\\y=z\end{matrix}$

(d) $\Rightarrow 4=\left(-\frac{1}{2}z\right)^2+(z)^2+3z^2+\left(-\frac{1}{2}z\right)z-2(z)z$

$4=\frac{7}{4}z^2$

$z=\pm\frac{4}{\sqrt{7}} \Rightarrow$ c.p.'s $\left(\frac{-2}{\sqrt{7}},\frac{4}{\sqrt{7}},\frac{4}{\sqrt{7}}\right)$, $\left(\frac{2}{\sqrt{7}},\frac{-4}{\sqrt{7}},\frac{-4}{\sqrt{7}}\right)$

Ex: Maximize $f(x,y) = x^2 + x + y^2 - y$ on $C = \{x^2+y^2=1\}$.
(Previously done without Lagrange multipliers)

$f(x,y) = x^2 + x + y^2 - y$, $g(x,y) = x^2+y^2$, both diff'bl.

$\nabla f = \begin{pmatrix} 2x+1 \\ 2y-1 \end{pmatrix}$ $\quad \nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ $= \vec{0}$ only at $\vec{x} = 0$ not on C!

$\nabla f = \lambda \nabla g \implies \begin{cases} 2x+1 = \lambda(2x) & ⓐ \\ 2y-1 = \lambda(2y) & ⓑ \end{cases}$

Constraint $\implies x^2+y^2=1$ ⓒ

ⓐ, ⓑ $\implies x, y \neq 0$

$\implies \lambda = 1 + \frac{1}{2x} = 1 - \frac{1}{2y}$

$\implies x = -y$

ⓒ $\implies (-y)^2 + y^2 = 1 \implies y = \pm\sqrt{1/2}$

$\implies$ c.p.'s $\left(-\sqrt{1/2}, \sqrt{1/2}\right)$, $\left(\sqrt{1/2}, -\sqrt{1/2}\right)$

Ex:) Find the point(s) on $x^2+4y^2+9z^2=36$ closest to the origin.

$f(x,y,z) = x^2+y^2+z^2$ $\quad$ $g(x,y,z) = x^2+4y^2+9z^2$, both diff'bl.

$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$ $\quad$ $\nabla g = \begin{pmatrix} 2x \\ 8y \\ 18z \end{pmatrix}$ $= \vec{0}$ only at $\vec{x}=\vec{0}$ not on S!

$\nabla f = \lambda \nabla g \implies \begin{cases} 2x = \lambda(2x) \\ 2y = \lambda(8y) \\ 2z = \lambda(18z) \end{cases}$

Constraint $\implies$ $x^2+4y^2+9z^2 = 36$

Tempting, but wrong, to see contradiction with λ's. But you can't cancel 0!

<u>Case 1</u>: $x \neq 0 \Rightarrow \lambda = 1 \implies y,z=0 \implies x = \pm 6$.

<u>Case 2</u>: $y \neq 0 \Rightarrow \lambda = \frac{1}{4} \implies x,z=0 \implies y = \pm 3$.

<u>Case 3</u>: $z \neq 0 \Rightarrow \lambda = \frac{1}{9} \implies x,y=0 \implies z = \pm 2$.

So we have c.p.'s $(\pm 6,0,0)$, $(0,\pm 3,0)$, $(0,0,\pm 2)$.