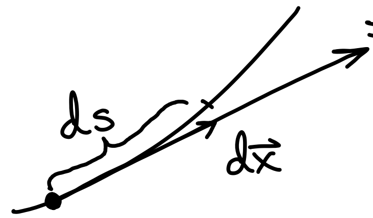
3.2 – Arclength

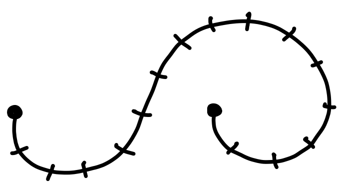
What is the length of the parametric curve $\vec{X}: [a,b] \to \mathbb{R}^n$?

$$d\vec{x} = \vec{x}'\,dt = \vec{v}\,dt = \vec{T}\,ds$$
$$ds = \|d\vec{x}\| = \|\vec{v}\|\,dt$$



$\vec{T}$ = unit vector in this direction $= \dfrac{\vec{X}'}{\|\vec{X}'\|} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Adding this up over the entire curve, we get



$$\text{length} = s = \int ds = \int_a^b \|\vec{v}\|\,dt$$

Ex:) What is the length of $\vec{x} = (t^2 - t,\, t^3)$ for $t \in [0,1]$?

$$S = \int_0^1 \|(2t-1,\, 3t^2)\|\,dt = \int_0^1 \sqrt{(2t-1)^2 + (3t^2)^2}\,dt$$

Arclength parameter

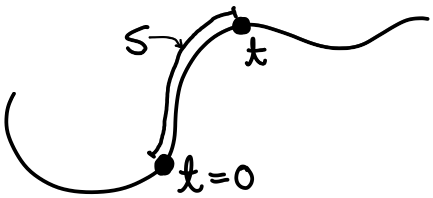
Recall that a curve has lots of parametrizations.

Ex:) $y = x^2$ is parametrized by both

$$\vec{x}(t) = (t, t^2) \quad \text{, and} \quad \vec{x}(q) = (q^3+q,\, (q^3+q)^2)$$

The parameters relate – in this case by $t = q^3 + q$.

If we have $\vec{x}(t)$, we can create a new parameter by



$$s(t) = \int_0^t \|\vec{x}'(\tau)\| \, d\tau$$

The arclength parametrization then is $\vec{x}(t(s)) = \vec{x}(s)$.

1. intrinsic to the curve;
2. unit speed:

$$\frac{d\vec{x}}{dt} = \frac{d\vec{x}}{ds}\frac{ds}{dt}$$

$$\left\|\frac{d\vec{x}}{dt}\right\| = \left\|\frac{d\vec{x}}{ds}\right\| \frac{ds}{dt}$$

these are equal, so this $=1$.

3. Unfortunately, it is rarely conveniently computable.

Ex:) Find the arclength parametrization of $\vec{x}(t) = \left(t - \frac{1}{3}t^3, t^2\right)$.

$$s(t) = \int_0^t \|\vec{x}'(\tau)\| \, d\tau = \int_0^t \sqrt{(1-\tau^2)^2 + (2\tau)^2} \, d\tau$$

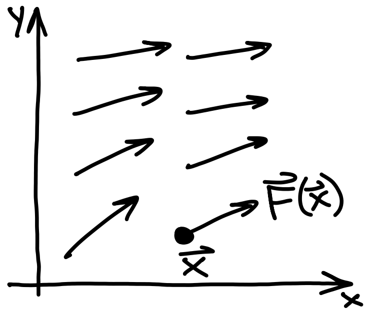
$$= \int_0^t \sqrt{(1 - 2\tau^2 + \tau^4) + (4\tau^2)} \, d\tau = \int_0^t \sqrt{(1+\tau^2)^2} \, d\tau$$

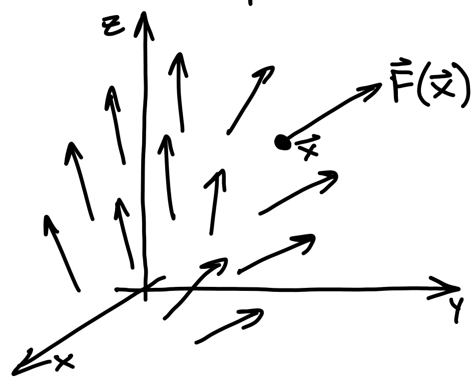
$$= \int_0^t 1 + \tau^2 \, d\tau = t + \frac{1}{3}t^3 = s$$

Hard to solve this for t ...

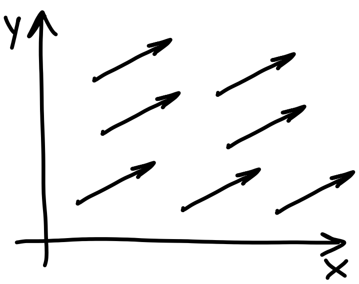
3.3 – Vector Fields

A vector field on $\mathbb{R}^n$ is a function $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, where the input is viewed as a point, the output as a vector.

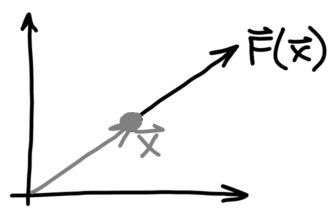


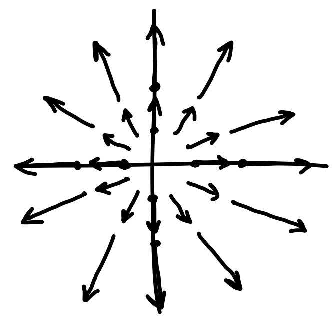


Ex:) $\vec{F}(x,y) = (2,1)$



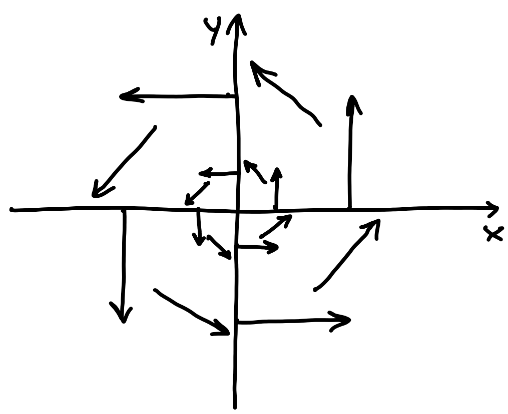
Ex:) $\vec{F}(x,y) = (x,y)$



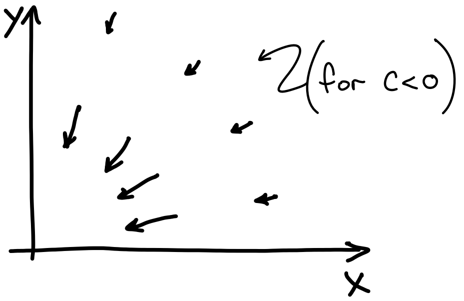


Ex:) $\vec{F}(x,y) = (-y,x)$

(Note $\vec{F}(\vec{x})$ is a quarter turn counterclockwise of $\vec{x}$.)



Ex:) $\vec{F}(\vec{x}) = c\dfrac{\vec{x}}{\|\vec{x}\|^3} = \dfrac{c}{\|\vec{x}\|^2}\left(\dfrac{\vec{x}}{\|\vec{x}\|}\right)$ ("radial inverse square")

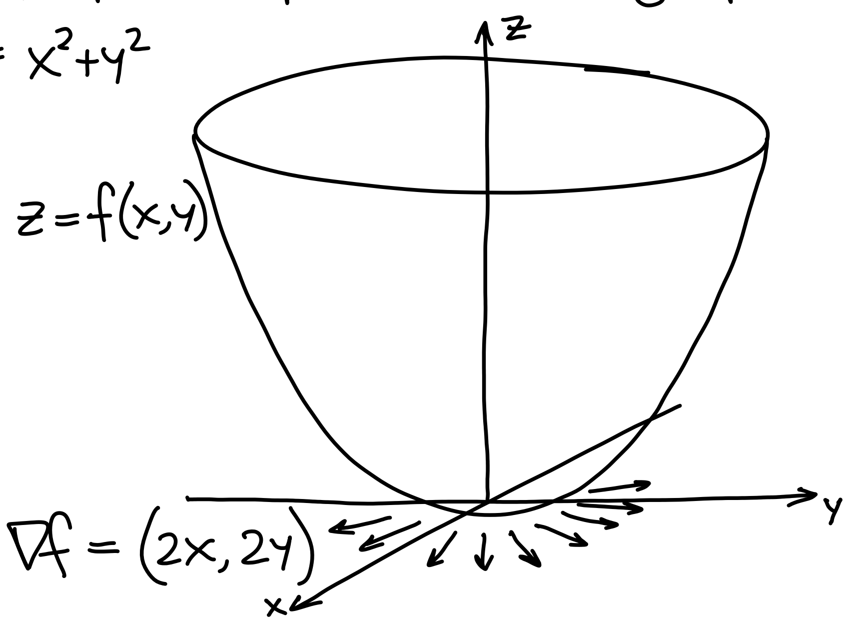


Gravitational, electric fields are of this form.

Gradients

$\vec{F} = \nabla f$ always points "uphill" on the graph of f.

Ex:) $f(x,y) = x^2+y^2$



$\nabla f = (2x, 2y)$

Terminology

$$f \xrightarrow{\nabla} \vec{F}$$

the "antigradient" of $\vec{F}$ (or "potential function")

the "gradient" of f

(In physics, usually $-(\text{potential energy}) \xrightarrow{\nabla} (\text{force})$
so often "potential function" refers to $-f$.)

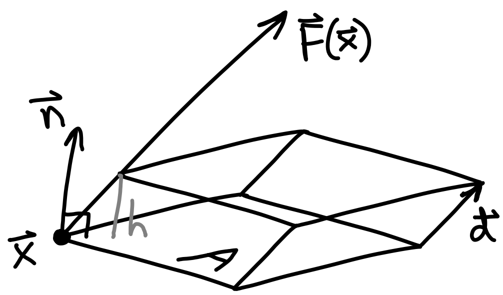
Instead of forces, a vector field could represent the flow of a fluid.

in $\mathbb{R}^3$:
- airflow
- ocean currents
- pollution
- atmospheric components

in $\mathbb{R}^2$:
- animal migrations
- surface water

<u>Ex:</u>) Suppose $\vec{F}(\vec{x})$ represents the velocity $\vec{v}$ of the fluid at the point $\vec{x}$.

Let A be a small, flat (normal to unit $\vec{n}$) area at $\vec{x}$. Then in time Δt the volume of fluid passing through A is



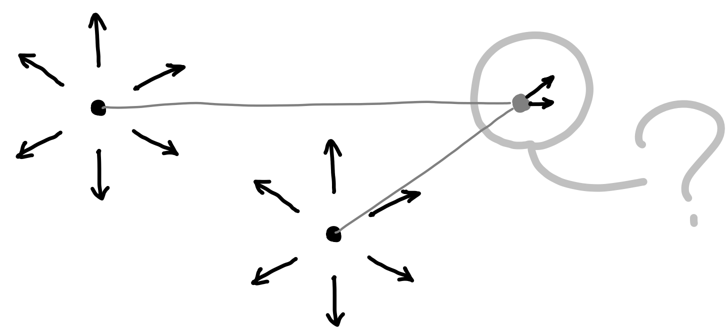
$$\begin{aligned}\Delta V &= hA \\ &= \left(\text{comp}_{\vec{n}}(\vec{d})\right) A \\ &= \left(\vec{d} \cdot \vec{n}\right) A \\ &= \Delta t \left(\vec{F} \cdot \vec{n}\right) A\end{aligned}$$

So the flow rate dV/dt is $(\vec{F} \cdot \vec{n})A$. This is called "flux", Φ.

<u>Ex:</u>) Suppose $\vec{F} = \delta \vec{v}$ instead. Then $\Phi = (\vec{F} \cdot \vec{n})A$ represents dm/dt, mass (instead of volume) per unit time.

For some fluid flows, it is hard to define $\vec{v}$, or $\delta\vec{v}$, but there is still dm/dt through flat areas.

Ex:) Consider the flow of photons generated by two point light sources.



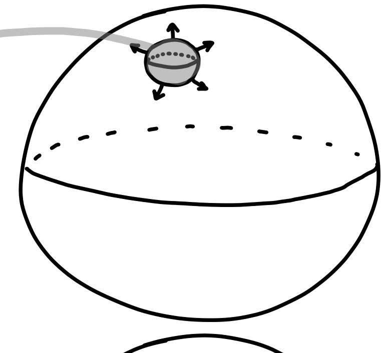
We can use flux to define a vector field representing the flow.

For every flat area (area $= A$, normal $= \vec{n}$) we have dm/dt; $\vec{F}$ is the vector field for which every

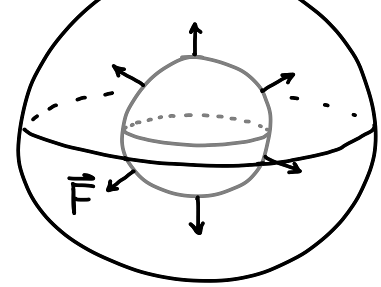
$$\frac{dm}{dt} = (\vec{F}\cdot\vec{n})A$$

(and similarly for other convenient surfaces.)

Ex:) We have a uniform ball of glowing gas, with $dm/dt = kV$



Overall $\vec{F}$ should be radial, so on a centered sphere we should have $\vec{F}\cdot\vec{n} = \|\vec{F}\|$. So



$$kV = \frac{dm}{dt} = (\vec{F}\cdot\vec{n})A = \|\vec{F}\|A$$

$$k\left(\tfrac{4}{3}\pi r^3\right) = \|\vec{F}\|\left(4\pi r^2\right) \Rightarrow \|\vec{F}\| = \frac{k}{3}r \Rightarrow \vec{F} = \frac{k}{3}\vec{x}$$

If $\vec{F}(\vec{x})=\vec{v}$, and a particle were dropped into the fluid and follow the current $\vec{v}$ to make a path $\vec{x}(t)$, then we would have

$$\vec{x}\,'(t) = \vec{v}(\vec{x}(t)) = \vec{F}(\vec{x}(t))$$

velocity of the particle

velocity of the fluid...

...where the particle is

Such a path is called a "flow line".

<u>Ex:</u>) $\vec{x}(t) = (\cos t, \sin t)$ is a flow line of $\vec{F}(x,y) = (-y, x)$.

It is important to be able to interpret vector fields <u>both</u> as forces <u>and</u> fluids.

Both are <u>critical</u> to future work in this course.

3.4 – Gradient, Divergence, Curl, and the Del Operator

Recall $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$. We can think of this as

$\nabla f = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} f$ and then write $\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$, the "del" operator.

This is a "symbolic vector", and ends up being convenient.

<u>Def:</u> The <u>divergence</u> of $\vec{F} = (F_1, F_2, F_3)$ is

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

(Similarly for vector fields in other $\mathbb{R}^n$.)

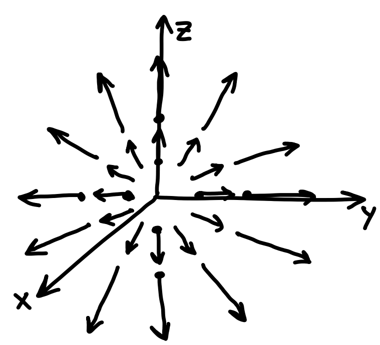
We don't yet have the tools to make sense of this formula.

It turns out though that $\nabla \cdot \vec{F}(\vec{x})$ represents the amount a fluid is flowing away from the vicinity of $\vec{x}$.

<u>Ex:</u> $\vec{F}(x,y,z) = (x,y,z)$

$\nabla \cdot \vec{F}(\vec{0}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

> 0, represents outward flow.

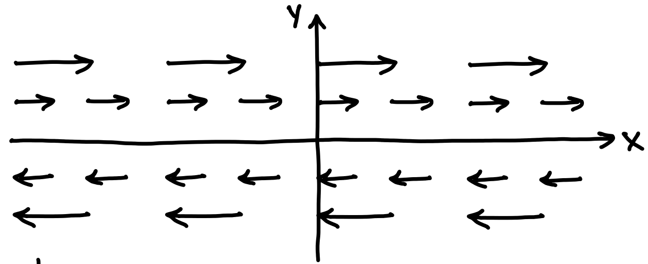


So at this moment, there must be some combination of

1. density decreasing there
2. fluid being created there.

Ex:) $\vec{F}(x,y) = (y, 0)$

$$\nabla \cdot \vec{F}(\vec{a}) = \frac{\partial y}{\partial x} + \frac{\partial 0}{\partial y} = 0$$



For a region around any point $\vec{a}$,
there's just as much fluid flowing in as out.

Def:) The <u>curl</u> of $\vec{F} = (F_1, F_2, F_3)$ is

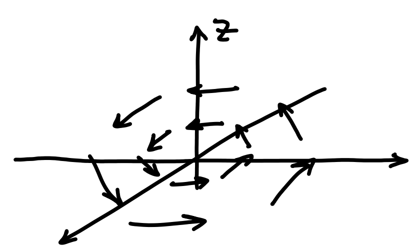
$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

We don't yet have the tools to make sense of this formula.

It turns out though that $\nabla \times \vec{F}(\vec{x})$ represents the amount (ccw) and axis the fluid "circulates" around near $\vec{x}$.

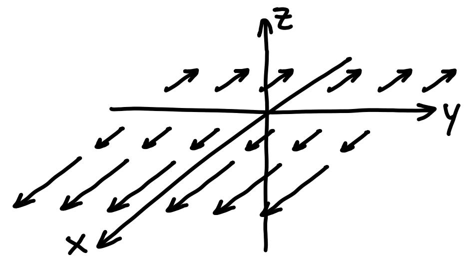
Ex:) $\vec{F}(x,y,z) = (-y, x, 0)$

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{pmatrix} = (0, 0, 2)$$



This is circulation around the vertical, ccw from above.

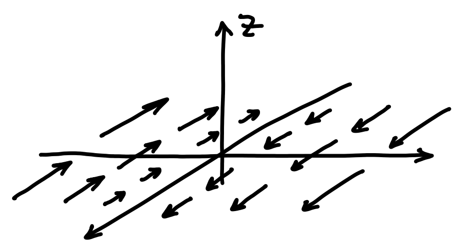
Ex:) $\vec{F}(x,y,z) = (x,0,0)$



$\nabla \times \vec{F} = \vec{0}$

No circulation.

Ex:) $F(x,y,z) = (y,0,0)$



$$\nabla \times \vec{F} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{pmatrix} = (0,0,-1)$$

The flow lines are all straight — but there is still circulation, around the vertical, ccw from below.

For reasons beyond this course, it is natural to <u>compose</u> these operators like this:

$$\{\text{fns}\} \xrightarrow{\nabla} \{\text{v.f.'s}\} \xrightarrow{\nabla\times} \{\text{v.f.'s}\} \xrightarrow{\nabla\cdot} \{\text{fns}\}$$

This is an exact* sequence (all "lifetimes" are 2).
(*- ignoring regularity issues, and the exception of constant functions.)

Ex: $\nabla \times (\nabla f) = \vec{0}$ (check!)

Ex: If $\nabla \cdot \vec{F} = 0$, then $\vec{F} = \nabla \times \vec{G}$ for some $\vec{G}$.
(this is hard to prove)

As this pattern suggests,

$\boxed{\text{there are lots of connections between these operators.}}$

We will see many of these in Chapters 6 and 7.