Chapter 1: Vectors

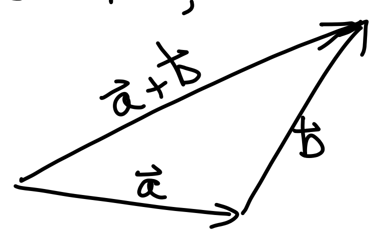
You have seen most of this already in linear algebra!

1.6 - Some n-dimensional Geometry

Be sure to review the linear algebra material that is most of this section!

Thm: (Triangle inequality) For all $\vec{a}, \vec{b} \in \mathbb{R}^n$,

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$



Thm: $A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ Then

$$A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$$

So: A matrix-vector product is:
a linear combination of the columns of the matrix, using the vector as the coefficients.

Def: A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> iff for all $a,b \in \mathbb{R}$, $\vec{v},\vec{w} \in \mathbb{R}^n$, we have

$$① \quad T(a\vec{v}+b\vec{w}) = aT(\vec{v}) + bT(\vec{w})$$

(**Equiv:** ② $T(a\vec{v}) = aT(\vec{v})$ and $T(\vec{v}+\vec{w}) = T(\vec{v})+T(\vec{w})$

Equiv: ③ $T(c_1\vec{v}_1 + \dots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \dots + c_nT(\vec{v}_n)$)

Such a function is sometimes called a "linear transformation".

Ex: Given $\vec{a} \in \mathbb{R}^n$ and $P(\vec{x}) = \vec{a}\cdot\vec{x}$, P is linear.

Ex: Given $\vec{a} \in \mathbb{R}^n$ and $C(\vec{x}) = \vec{a}\times\vec{x}$, C is linear.

Ex: Given $\vec{r}_1, \dots, \vec{r}_{n-1} \in \mathbb{R}^n$, then D (below) is linear.

$$D(\vec{x}) = \det\begin{pmatrix} \text{---} & \vec{r}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_{n-1} & \text{---} \\ \text{---} & \vec{x} & \text{---} \end{pmatrix}$$

Thm:) $(T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation$)$

$$\Updownarrow$$

$(\exists$ a matrix A with $T(\vec{x}) = A\vec{x}$ for all $\vec{x}.)$

Pf:) $(\Uparrow)$ Direct computation.

$(\Downarrow)$ Let $\vec{x} = (x_1, \dots, x_n)$. Then by linearity we have

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + \dots x_n T(\vec{e}_n) \end{aligned}$$

We can then choose $\vec{a}_i = T(\vec{e}_i)$ and

$$A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{pmatrix}$$

and then rewrite as

$$\begin{aligned} T(\vec{x}) &= x_1 T(\vec{e}_1) + \dots + x_n T(\vec{e}_n) \\ &= x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = A\vec{x} \end{aligned}$$

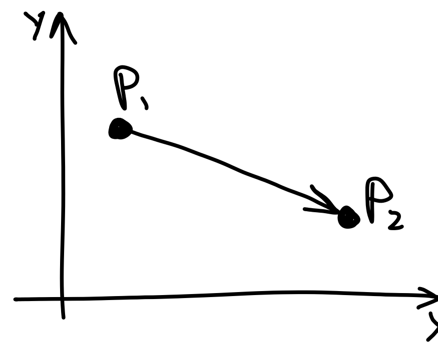
Thm:) Matrix multiplication corresponds to composition of linear transformations. That is,

$$S(\vec{x}) = A\vec{x}, \ T(\vec{x}) = B\vec{x} \implies (S \circ T)(\vec{x}) = (AB)\vec{x}$$

1.1 – Vectors in Two and Three Dimensions

The book uses bold ("$\mathbf{v}$") to indicate a vector.
I use a vector hat ("$\vec{v}$") because it is easier to write!

The book represents this vector as "$\overrightarrow{P_1P_2}$".



I prefer to write it as "$\vec{P}_2 - \vec{P}_1$".

- shows the implicit algebra!
- recall natural correspondence between points & vectors!

1.2 – More About Vectors

A "parametrization" gives position as a function of a single parameter (say, t). If you think of t as time, then the parametrization "draws" the "parametric curve".

Ex: $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. Notice that (x,y) is always on $x^2+y^2=1$.

Ex: $\vec{x}(t) = (t-1, 3t+4)$.

① $t = x+1 \Rightarrow y = 3t+4 = 3(x+1)+4$
$\Rightarrow y = 3x+7$

Do we see a relationship between $x = t-1$ and $y = 3t+4$?

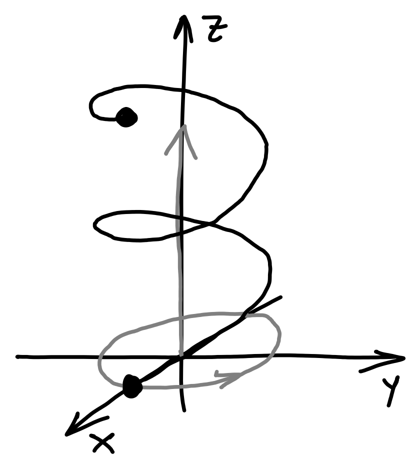
②
$$x = t-1$$
$$3x = 3t-3$$
$$3x+7 = 3t+4 = y \Rightarrow y = 3x+7$$

Ex: $\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

Curve in $\mathbb{R}^3$; but an equation would be a 'surface'!

Obs: ① projection to xy-plane is moving in a circle.

② projection to z-axis is moving upward.



Combining these, we see we have a helix.

What if you know the curve and want to "parametrize"?

Graph parametrization: If you have a graph $y = f(x)$, let t be the input variable!

Ex: We can parametrize $y = x^2$ with $\vec{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$.

Ex:) What if $x e^{y^2} + 3y = 2$?

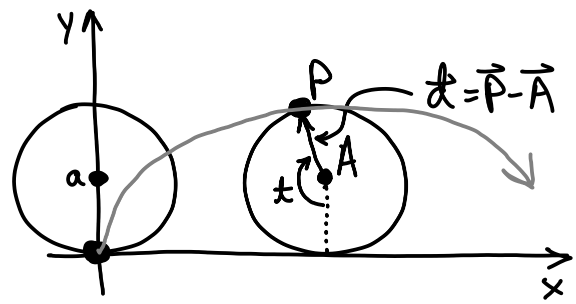
Write as $x = e^{-y^2}(2-3y)$! Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t^2}(2-3t) \\ t \end{pmatrix}$$

Or, try using vectors...

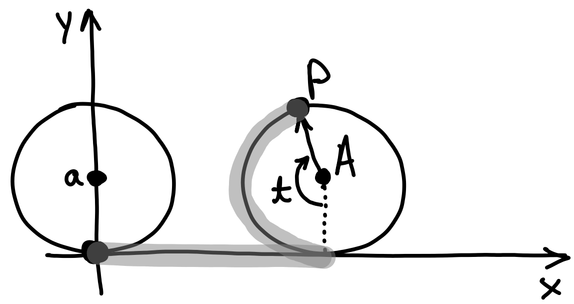
Ex:) Wheel of radius a rolling to the right :

$$\vec{P} = \vec{A} + \vec{d}$$



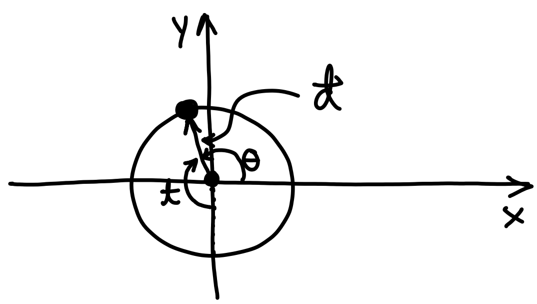
Rubber = Road !

$$\Rightarrow \vec{A} = \begin{pmatrix} at \\ a \end{pmatrix}$$



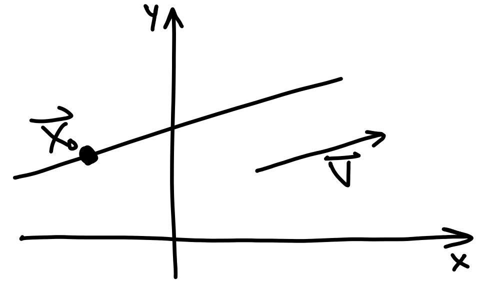
$$\theta = \frac{3\pi}{2} - t, \quad \vec{d} = \begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}$$

$$\Rightarrow \vec{d} = \begin{pmatrix} -a\sin t \\ -a\cos t \end{pmatrix}$$

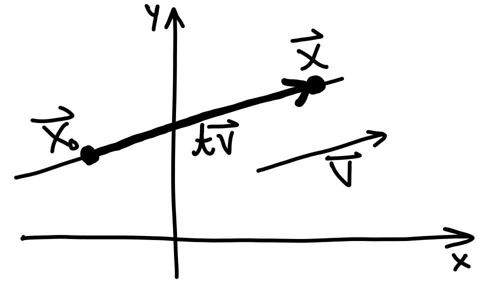


So $\vec{P} = \begin{pmatrix} at \\ a \end{pmatrix} + \begin{pmatrix} -a\sin t \\ -a\cos t \end{pmatrix}$.

Ex:) Say we have a line through $\vec{x}_0$ parallel to $\vec{v}$...



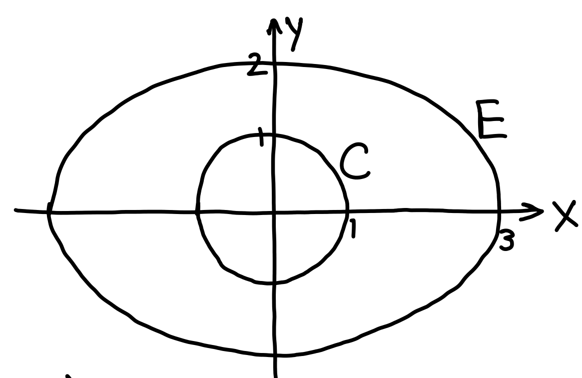
Then $\vec{x} = \vec{x}_0 + t\vec{v}$ draws the line.



Or you can deform an existing parametrization.

Ex:) Parametrize the ellipse with equation $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

$\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ makes the circle C.



Stretch C by

- x3 in the x-direction
- x2 in the y-direction

$\Longrightarrow$ E is parametrized by $\begin{pmatrix} 3\cos t \\ 2\sin t \end{pmatrix}$.

Notice that a <u>parametrization</u> is a "point generator":

$$t \rightarrow \boxed{\text{param.}} \rightarrow (x,y) \text{ on the curve}$$

But an <u>equation</u> is a "point tester":

$$\underset{\text{(on the curve?)}}{(x,y)} \rightarrow \boxed{\text{eqn.}} \rightarrow \text{yes or no}$$

<u>Ex:</u> What point(s) on $\vec{x}(t) = (t^2-1,\ t+1,\ t^2+t)$ are on the plane with equation $2x+y-z=0$?

Strategy: <u>Test</u> the <u>generated</u> points!

$$t \xrightarrow{\text{param}} \begin{matrix} x = t^2-1 \\ y = t+1 \\ z = t^2+t \end{matrix} \xrightarrow{2x+y-z=0} 2(t^2-1)+(t+1)-(t^2+t) \overset{?}{=} 0$$

$$\Leftrightarrow t^2-1 = 0 \quad \Rightarrow t = \pm 1$$

Reminder: The "point tester" for a line in $\mathbb{R}^3$ is the "symmetric equations":

$$\exists t) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Updownarrow$$

$$\exists t) \quad t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

$$\Updownarrow$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

1.3 - The Dot Product

Recall from linear algebra:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n$$

Review the many properties in the book!

Recall also, this is the "model" for the idea of inner products.

NB: The book uses "perpendicular" and "orthogonal" interchangeably.

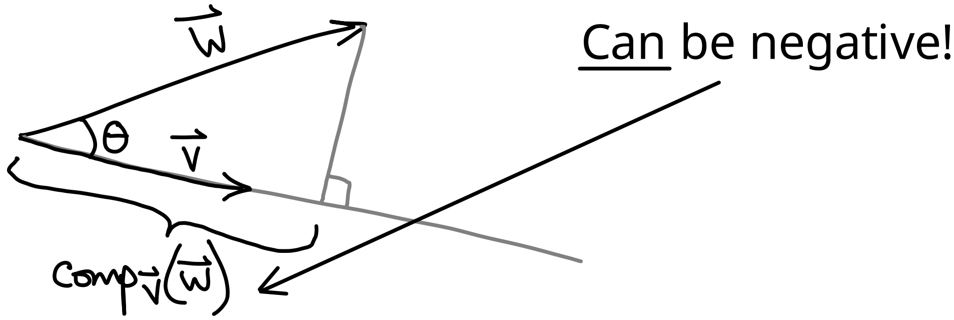
I prefer to use

- "perpendicular" as a geometric idea (nonzero vectors only!),
- "orthogonal" as an algebraic idea (whenever $\vec{u} \cdot \vec{v} = 0$).

Thm) For any $\vec{v} \neq \vec{0}$, the unique unit vector pointing in the same direction as $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

The <u>component</u> of $\vec{w}$ in the direction of $\vec{v}$ is as pictured below.



Trig then gives us

$$\cos\theta = \frac{\text{comp}_{\vec{v}}(\vec{w})}{\|\vec{w}\|} \implies \text{comp}_{\vec{v}}(\vec{w}) = \|\vec{w}\| \cos\theta$$

$$= \frac{\|\vec{v}\| \, \|\vec{w}\| \cos\theta}{\|\vec{v}\|}$$

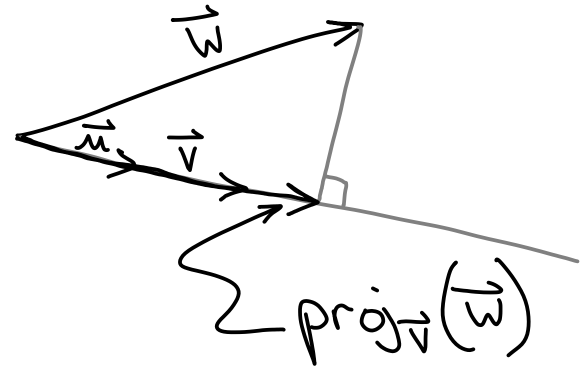
$$= \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|}$$

Better, with $\vec{u} = \vec{v}/\|\vec{v}\|$, we can write

$$\text{comp}_{\vec{v}}(\vec{w}) = \vec{w} \cdot \vec{u}$$

$$\boxed{\text{Components are dot products with unit vectors.}}$$

The projection of $\vec{w}$ onto $\vec{v}$ is the vector in the $\vec{v}$-direction in the amount of $\text{comp}_{\vec{v}}(\vec{w})$, as shown.



So we can compute it as:

$$\text{proj}_{\vec{v}}(\vec{w}) = \text{comp}_{\vec{v}}(\vec{w})\,\vec{u}$$
$$= (\vec{w}\cdot\vec{u})\,\vec{u}$$

(The book writes this instead as

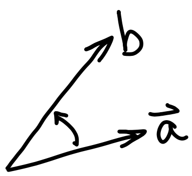
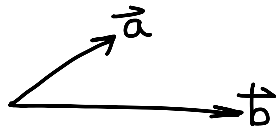
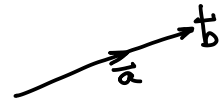
$$\text{proj}_{\vec{v}}(\vec{w}) = (\vec{w}\cdot\vec{u})\,\vec{u}$$
$$= \left(\vec{w}\cdot\frac{\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$
$$= \frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v} = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}\;)$$

Vector algebra can help with proofs in plane geometry. See two nice examples in the book.

1.4 – The Cross Product

Geometric order of a list

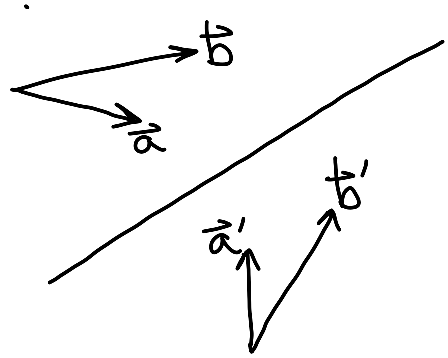
An ordered list $(\vec{a}, \vec{b})$ of 2 vectors in $\mathbb{R}^2$ is either:

counterclockwise	clockwise	(linearly) dependent
		

Some properties:

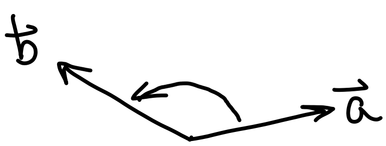
① ccw and cw are "mirror images".

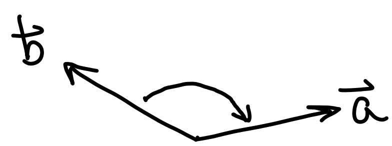
$(\vec{a}, \vec{b})$ ccw $\Longleftrightarrow$ $(\vec{a}', \vec{b}')$ cw



② "Trading" positions in the list changes order.

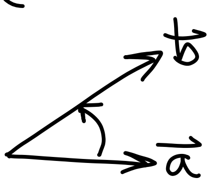
$(\vec{a}, \vec{b})$ ccw $\Longleftrightarrow$ $(\vec{b}, \vec{a})$ cw

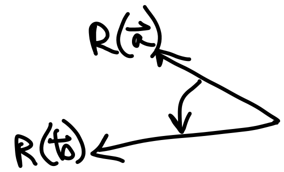




③ Order is independent of rotations.

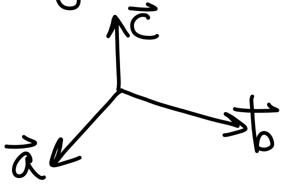
$(\vec{a}, \vec{b})$ ccw $\Longleftrightarrow$ $(R(\vec{a}), R(\vec{b}))$ ccw



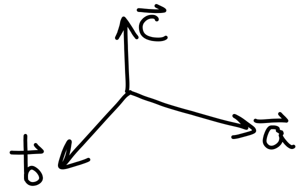


An ordered list $(\vec{a}, \vec{b}, \vec{c})$ of 3 vectors in $\mathbb{R}^3$ is either:

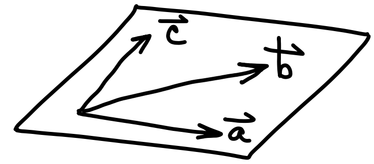
<u>right handed</u>



<u>left handed</u>



<u>dependent</u>



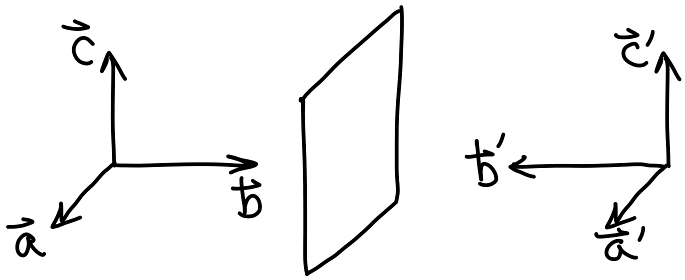
<u>Some properties:</u>

① Right handed and left handed are "mirror images".

$(\vec{a}, \vec{b}, \vec{c})$ RHO

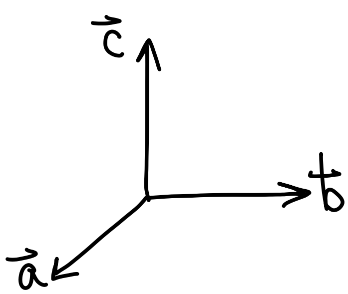
$\Updownarrow$

$(\vec{a}', \vec{b}', \vec{c}')$ LHO



② Trading positions of 2 vectors in the list changes the order.

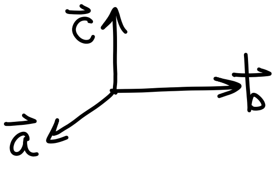
$(\vec{a}, \vec{b}, \vec{c})$ RHO

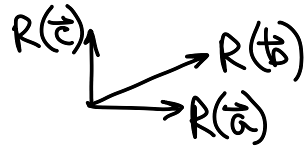


$(\vec{b}, \vec{a}, \vec{c})$ LHO

③ Order is independent of rotations.

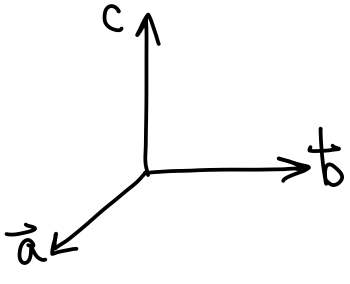
$(\vec{a}, \vec{b}, \vec{c})$ RHO $\Longleftrightarrow$ $(R(\vec{a}), R(\vec{b}), R(\vec{c}))$ RHO





④ "Cycling" a list preserves order.

$(\vec{a},\vec{b},\vec{c})$ RHO

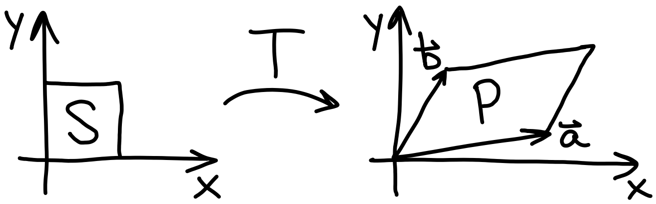


$(\vec{b},\vec{c},\vec{a})$ RHO

Two theorems you may have seen in linear algebra:

<u>Thm:</u>) From a list $(\vec{a},\vec{b})$ in $\mathbb{R}^2$ we make a matrix A, the linear transformation $T(\vec{x})=A\vec{x}$, and the parallelogram $P=T(S)$ with edge vectors $\vec{a}$ and $\vec{b}$.

$$A=\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$



Then:

① $|\det A|$ = area of P

② sign of $\det A$ indicates order of $(\vec{a},\vec{b})$ <u>and</u> whether T "inverts" the plane:

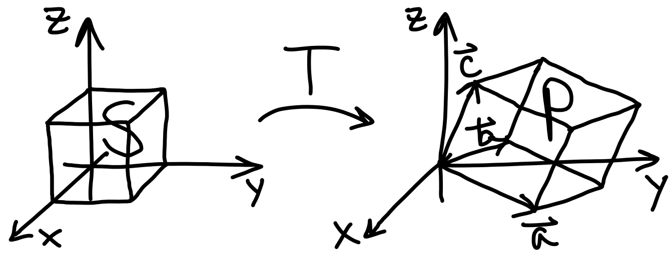
$\det A>0$: $(\vec{a},\vec{b})$ ccw, plane <u>not</u> inverted, all orders preserved

$\det A<0$: $(\vec{a},\vec{b})$ cw, plane <u>is</u> inverted, all orders reverse.

$\det A=0$: $(\vec{a},\vec{b})$ dependent, plane is squished.

Thm: From a list $(\vec{a},\vec{b},\vec{c})$ in $\mathbb{R}^3$ we make a matrix A, the linear transformation $T(\vec{x})=A\vec{x}$, and the parallelepiped $P=T(S)$ with edge vectors $\vec{a},\vec{b},\vec{c}$.

$$A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$



Then:

① $|\det A|$ = volume of P

② sign of $\det A$ indicates order of $(\vec{a},\vec{b},\vec{c})$ and whether T "inverts" $\mathbb{R}^3$:

$\det A>0$: $(\vec{a},\vec{b},\vec{c})$ RHO, $\mathbb{R}^3$ not inverted, all orders preserved

$\det A<0$: $(\vec{a},\vec{b},\vec{c})$ LHO, $\mathbb{R}^3$ is inverted, all orders reverse.

$\det A=0$: $(\vec{a},\vec{b},\vec{c})$ dependent, $\mathbb{R}^3$ is squished.

(What follows is different from the book – but equivalent.)

Def: The cross product $\vec{a}\times\vec{b}$ of two vectors in $\mathbb{R}^3$ is

$$\vec{a}\times\vec{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ -(a_1b_3-a_3b_1) \\ a_1b_2-a_2b_1 \end{pmatrix}=\det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

(symbolic!)

Notice that

$$\boxed{\vec{c}\cdot(\vec{a}\times\vec{b})} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \boxed{\det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}$$

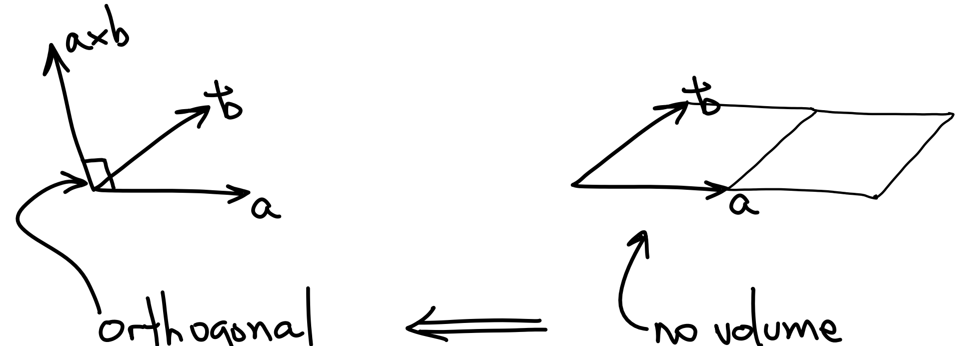
This "triple product" formula is a powerful connection between dot product and determinant (which we understand), and cross product (which we don't yet).

<u>Thm:</u> $\vec{a}\times\vec{b}$ is $\perp$ to <u>both</u> $\vec{a}$ and $\vec{b}$.

<u>Pf:</u> Compute $\vec{a}\cdot(\vec{a}\times\vec{b})$ and $\vec{b}\cdot(\vec{a}\times\vec{b})$ directly.

Using the triple product for example:

$$\vec{a}\cdot(\vec{a}\times\vec{b}) = \det\begin{pmatrix} \text{---}\ \vec{a}\ \text{---} \\ \text{---}\ \vec{a}\ \text{---} \\ \text{---}\ \vec{b}\ \text{---} \end{pmatrix} = 0.$$



<u>Thm:</u> $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

<u>Pf:</u> The three determinants are equal by transpositions.

Or – magnitudes computed as volume of the same parallelepiped, signs equal because the lists have the same handedness.

Thm: $\|\vec{a}\times\vec{b}\| = \text{area}\left(\parallel(\vec{a},\vec{b})\right)$

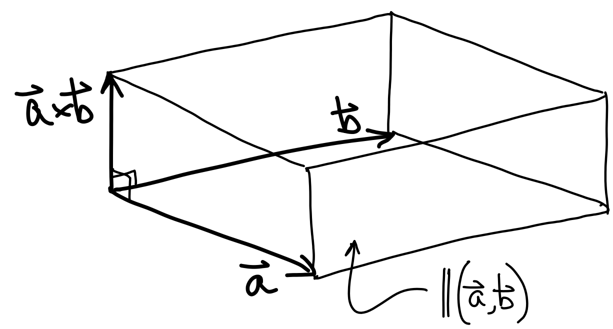
Pf: Consider

$$\det\begin{pmatrix} \text{---}\ \vec{a}\times\vec{b}\ \text{---} \\ \text{---}\ \vec{a}\ \text{---} \\ \text{---}\ \vec{b}\ \text{---} \end{pmatrix}$$

① $=$ triple product $= (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = \|\vec{a}\times\vec{b}\|^2$

② $= \text{volume}\left(\parallel(\vec{a}\times\vec{b},\vec{a},\vec{b})\right)$

$= \left(\|\vec{a}\times\vec{b}\|\right)\left(\text{area}\left(\parallel(\vec{a},\vec{b})\right)\right)$



Cancelling the common factor gives the result.

Thm: $\vec{a}\times\vec{b} = \vec{0} \iff (\vec{a},\vec{b})$ is linearly dependent

Pf: Both correspond to $\text{area}\left(\parallel(a,b)\right) = 0$.

Thm: $(\vec{a},\vec{b},\vec{a}\times\vec{b})$ is <u>never</u> in left hand order.

Pf: $\det\begin{pmatrix} \text{---}\ \vec{a}\times\vec{b}\ \text{---} \\ \text{---}\ \vec{a}\ \text{---} \\ \text{---}\ \vec{b}\ \text{---} \end{pmatrix} = \|\vec{a}\times\vec{b}\|^2$ is never negative, so $(\vec{a}\times\vec{b},\vec{a},\vec{b})$ is never left hand order. By "cycling" we get the result.

We now have a purely geometric description of the cross product $\vec{a} \times \vec{b}$, as the unique vector with

1. $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$
2. $\|\vec{a} \times \vec{b}\| = \text{area}(\|(\vec{a}, \vec{b}))$
3. $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ is not left handed.

(The book uses this as the definition.)

<u>Thm:</u> $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

<u>Pf:</u> Same magnitude, $\text{area}(\|(\vec{a}, \vec{b}))$. If not zero, then they point in opposite directions by handedness.

<u>NB also:</u>

1. $T(\vec{x}) = \vec{a} \times \vec{x}$ is linear (and likewise if you switch the order).

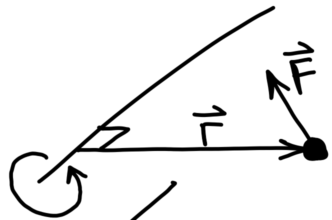
That is:
$$\vec{a} \times (c_1 \vec{x}_1 + c_2 \vec{x}_2) = c_1 (\vec{a} \times \vec{x}_1) + c_2 (\vec{a} \times \vec{x}_2)$$

Similarly $T(\vec{x}) = \vec{x} \times \vec{a}$ is linear:
$$(c_1 \vec{x}_1 + c_2 \vec{x}_2) \times \vec{a} = c_1 (\vec{x}_1 \times \vec{a}) + c_2 (\vec{x}_2 \times \vec{a})$$

2. Cross product is <u>not</u> associative.
$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Applications

Torque: is a measure of how hard you are turning something around an axis.

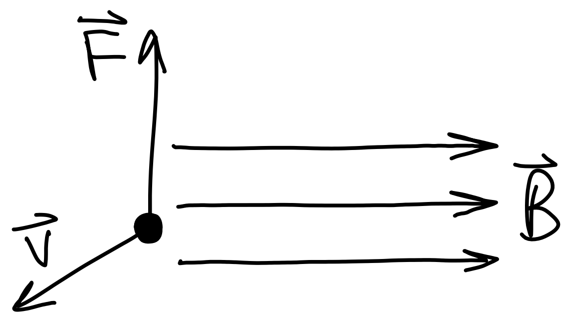


$$\vec{\tau} = \vec{r} \times \vec{F}$$

Movement through a magnetic field:

A particle of charge q moving through a magnetic field $\vec{B}$ with velocity $\vec{v}$ experiences a force

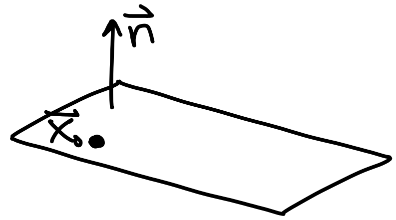
$$\vec{F} = q\vec{v} \times B$$



1.5 – Equations for Planes, Distance Problems

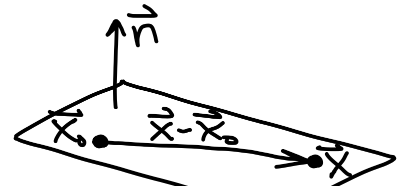
Suppose we know about a plane P:

$$\text{① } P \perp \vec{n}$$
$$\text{② } P \ni \vec{x}_0$$



Then from geometry we see that

$$\vec{x} \in P \iff \vec{x} - \vec{x}_0 \parallel P$$
$$\iff (\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$$

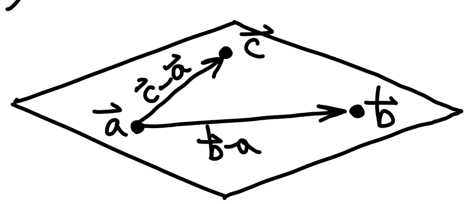


So a general equation for such a plane P is
$$\boxed{\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0}$$

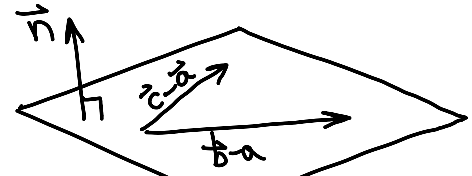
Writing this as
$$ax + by + cz = d$$
note that the coefficients a, b, c are the components of $\vec{n} = (a, b, c)$.

Ex: Find the equation of the plane containing $\vec{a} = (1,0,2)$, $\vec{b} = (3,2,4)$, $\vec{c} = (1,2,3)$.

Note that $\vec{b} - \vec{a}$, $\vec{c} - \vec{a} \parallel P$.



So we can use $\vec{n} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$, and let $\vec{x}_0$ be either $\vec{a}, \vec{b}$, or $\vec{c}$.



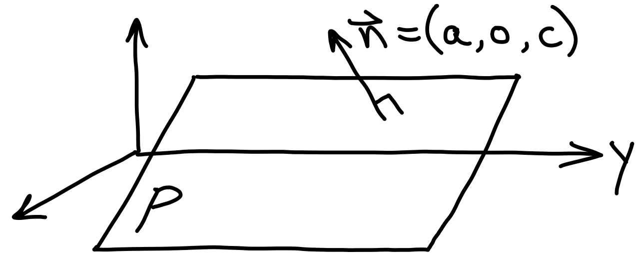
Ex: If there are no y's in the equation for a plane P, then P is parallel to the y-axis. Why?

$ax + 0y + cz = d$

$\Longrightarrow \vec{n} = (a, 0, c)$

$\Longrightarrow \vec{n} \perp$ y-axis

$\Longrightarrow P \parallel$ y-axis

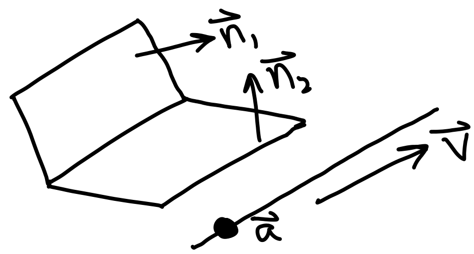


Ex: Parametrize the line L through $\vec{a} = (1,2,3)$ that is parallel to both $x + y = 3$ and $x - 3y + z = 0$.

L is $\perp$ to both normals $\vec{n}_1 = (1,1,0)$, $\vec{n}_2 = (1,-3,1)$.

So $\vec{v} = \vec{n}_1 \times \vec{n}_2$ is $\parallel$ to L.

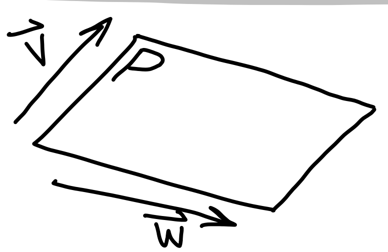
Then $L = \{\vec{a} + t\vec{v}\}$.



Warning: Always think through carefully any "transitivity" arguments about perpendicular / parallel...

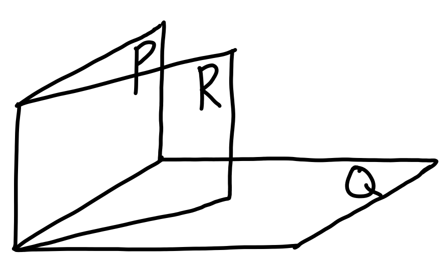
Ex: $\vec{v} \parallel P$ and $P \parallel \vec{w}$

... but $\vec{v} \nparallel \vec{w}$!



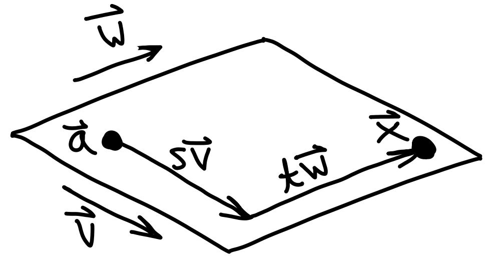
Ex: $P \perp Q$, $Q \perp R$

... but $P \not\perp R$!



How can we parametrize ("point generator") a plane?
We need <u>two</u> parameters:

$$\vec{x} = \vec{a} + s\vec{v} + t\vec{w}$$

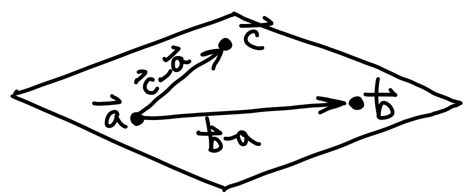


generates the plane through $\vec{a}$ parallel to $\vec{v}, \vec{w}$.

<u>Ex:</u> Find a parametrization of the plane containing $\vec{a} = (1,0,2)$, $\vec{b} = (3,2,4)$, $\vec{c} = (1,2,3)$.

Again we have

$$\vec{b} - \vec{a},\ \vec{c} - \vec{a} \parallel P$$



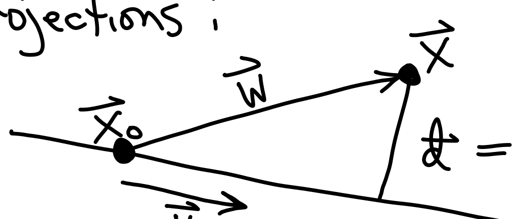
So we can parametrize by

$$\vec{x} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$$

<u>Distance problems:</u>

From a point to a line:

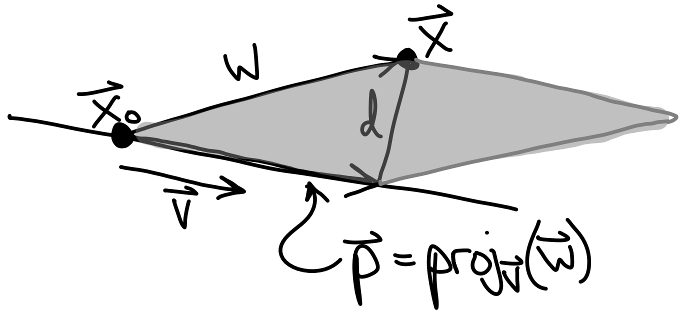
① Use projections:



$$\vec{d} = \vec{w} - \mathrm{proj}_{\vec{v}}(\vec{w})$$

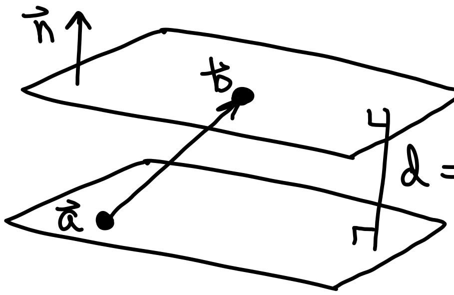
$$\text{distance} = \|\vec{d}\|$$

② Use area and cross products:



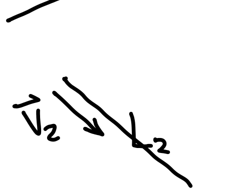
$$(d)\|\vec{p}\| = \text{area} = \|\vec{p}\times\vec{w}\|$$

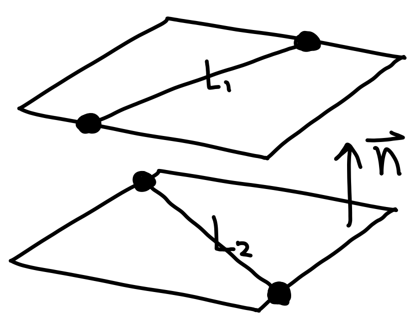
Between parallel planes: Use components in the $\vec{n}$ direction.



$$d = \left|\mathrm{comp}_{\vec{n}}\left(\vec{b}-\vec{a}\right)\right|$$

Between skew lines: identify parallel planes containing them.





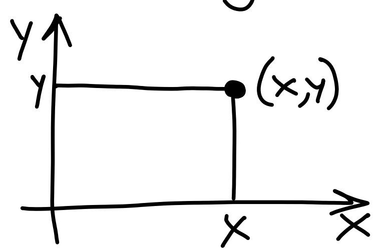
$$\vec{n} = \vec{v}_1 \times \vec{v}_2$$

1.7 – New Coordinate Systems

The "usual" coordinate system is "rectangular coordinates".

① Unique:

$\{\text{coords } x, y\} \longleftrightarrow \{\text{points in } \mathbb{R}^2\}$



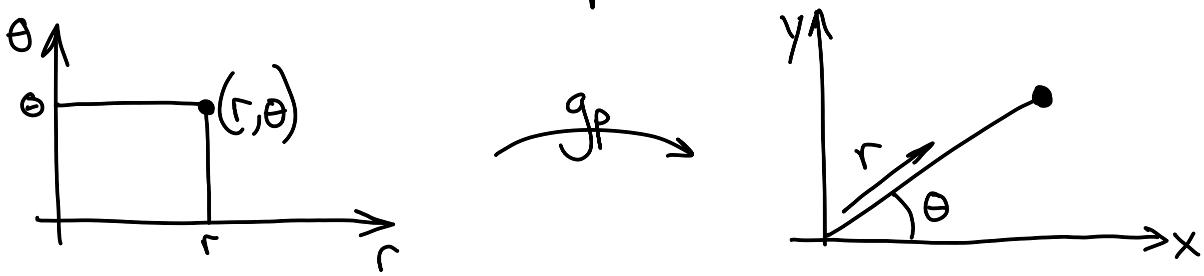
② Algebraically convenient:

$$(x,y) = x\vec{e}_1 + y\vec{e}_2 \quad ; \quad a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix} \; ; \; \ldots$$

None of these hold for the following coordinate systems!

Polar Coordinates

Think of r, θ as the inputs to a function



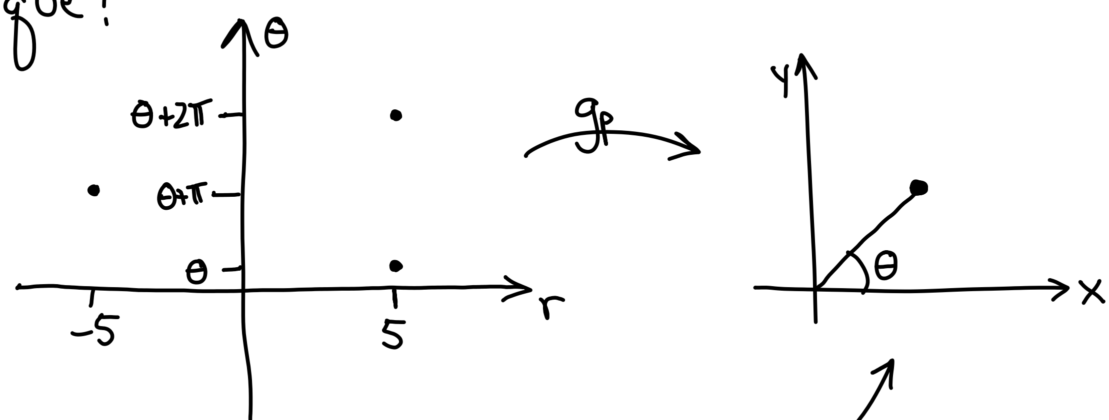
θ: defines a direction from the origin.

r: defines "how far" in that direction (if $r<0$, just go backward!)

The function is given by

$$g(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$$

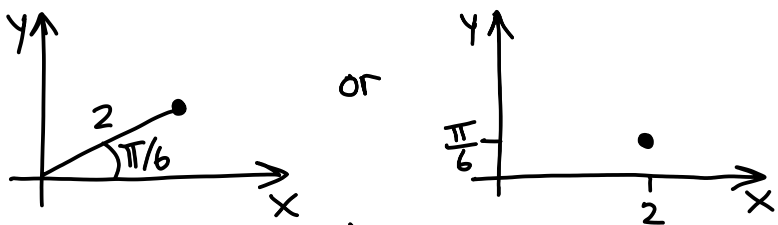
Not unique!



(these are all "polar coordinates" of this same point

NB, very often (our textbook too!) polar coordinates are given as an ordered pair, with no specific indication of being polar.

Ex:) Where is $(2, \pi/6)$?



?

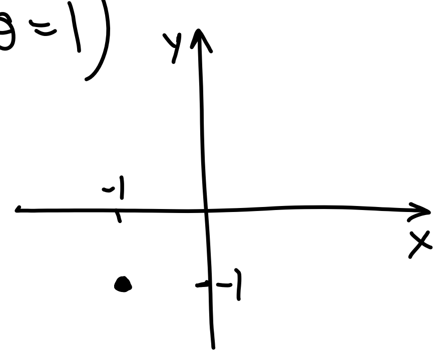
Clarify if feasible! E.g., $(r,\theta) = (2, \pi/6)$.

Converting back: Use $r^2 = x^2 + y^2$, $\tan\theta = y/x$, and geometry. Cannot solve these in general for r, θ!

Ex:) $(x,y) = (-1,-1)$ $\left(\Rightarrow r^2 = 2,\ \tan\theta = 1\right)$

$\Rightarrow$ $r = \sqrt{2}$ or $-\sqrt{2}$

$\theta = \frac{\pi}{4} + 2\pi n$ or $\frac{5\pi}{4} + 2\pi n$

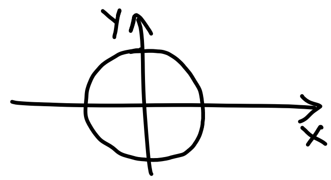


But NB, $(r,\theta) = (\sqrt{2}, \pi/4)$ is wrong!

Equations in polar coordinates

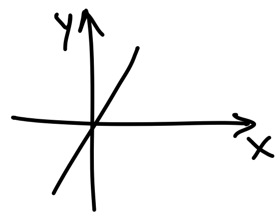
Ex: $r = 3$

$\Rightarrow r^2 = 9 \Rightarrow x^2 + y^2 = 9$



Ex: $\theta = \pi/3$

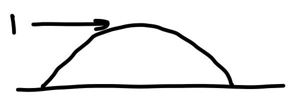
$\Rightarrow \tan\theta = \sqrt{3} \Rightarrow \frac{y}{x} = \sqrt{3}$



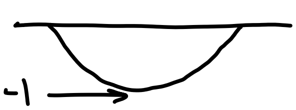
Ex: $r = \sin 2\theta$

Consider over ranges of angles.

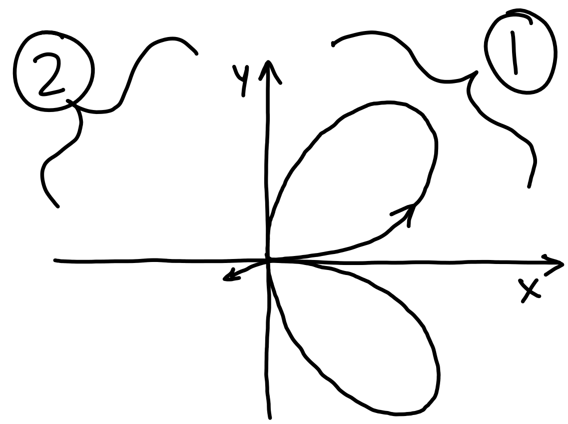
① $\theta \in [0, \pi/2]$:



② $\theta \in [\pi/2, \pi]$:



⋮

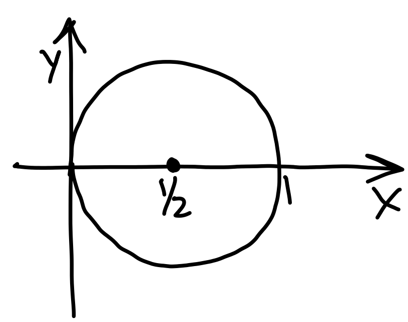


Ex: $r = \cos\theta$

$\Rightarrow r^2 = r\cos\theta \Rightarrow x^2 + y^2 = x$

$\Rightarrow \left(x^2 - x + \frac{1}{4}\right) + y^2 = \frac{1}{4}$

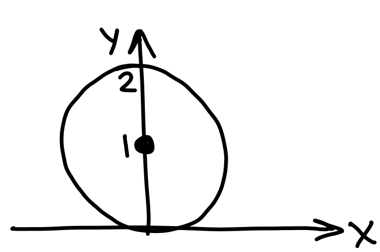
$\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$



Ex: $x^2 + (y-1)^2 = 1$

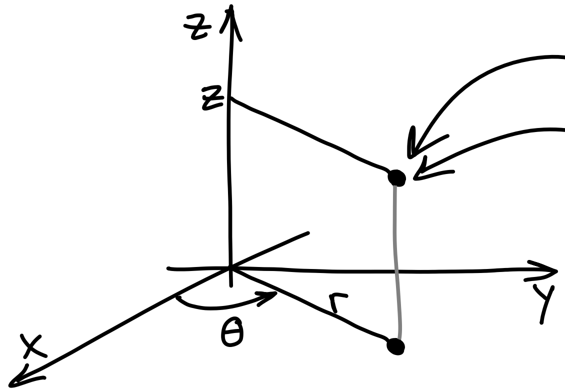
$x^2 + y^2 - 2y + 1 = 1$

$r^2 - 2(r\sin\theta) = 0 \Rightarrow r = 2\sin\theta$ (or $r = 0$)



Cylindrical coordinates

Polar coordinates for the xy-projection; leave the z.



rect coords (x,y,z)

cyl. coords (r,θ,z)

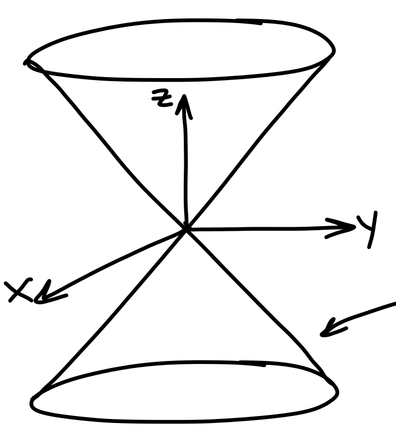
$x = r\cos\theta \qquad x^2+y^2=r^2$

$y = r\sin\theta \qquad \tan\theta = y/x$

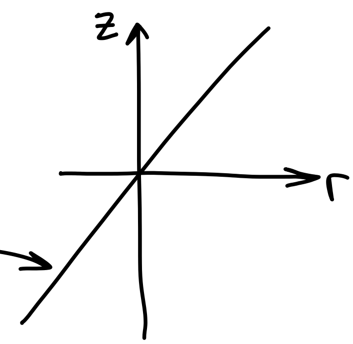
Ex:) $r=c$ is always a cylinder. (Thus the name!)

Ex:) Spheres $x^2+y^2+z^2=c^2$ have cylindrical equations $r^2+z^2=c^2$

Ex:) $z=kr$ is a cone.



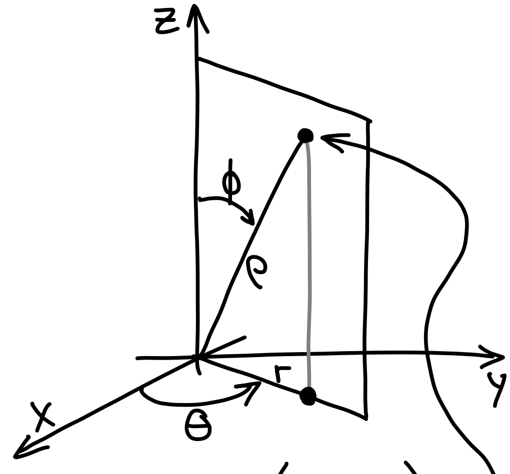
is a rotation of



Of course r can be negative!

NB- be careful using "r" as a radius (of a given circle, sphere, cylinder,...). Confusion of variables!

Spherical coordinates

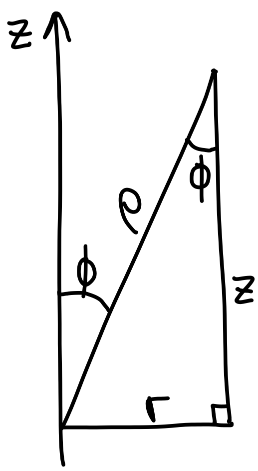


sph. coords (ρ, ϕ, θ)

θ: defines a half plane "hinged" on the z-axis.

ϕ (phi ("fee")): defines an angle from the +z-axis toward the θ half plane (or, away if $\phi < 0$).

ρ (rho ("row")): defines "how far" in that direction (if $\rho < 0$, backward!)



$\Rightarrow$ $r = \rho \sin\phi$
$z = \rho\cos\phi$

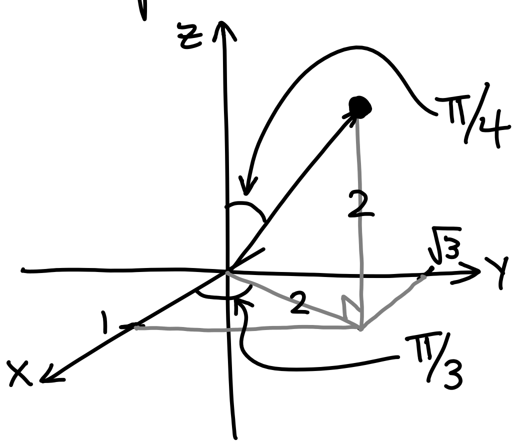
$\Rightarrow$ $x = (\rho\sin\phi)\cos\theta$
$y = (\rho\sin\phi)\sin\theta$
$z = \rho\cos\phi$

Ex: $x^2+y^2+z^2 = \rho^2$, so $\rho = c$ is always a sphere centered at the origin. (Thus the name!)

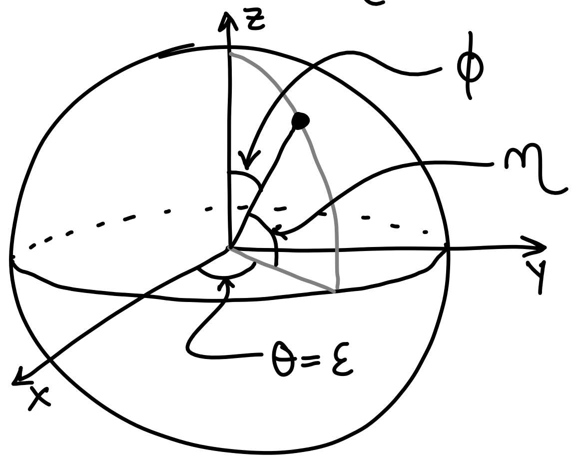
Ex: There are two ways to write $\vec{p} \in \mathbb{R}^2$ in polar coords, and four ways to write $\vec{p} \in \mathbb{R}^3$ in spherical coords!
Consider $\vec{p} = (1, \sqrt{3}, 2)$:

$\theta = \frac{\pi}{3}$ or $\frac{\pi}{3} + \pi$

For each, 2 options for ϕ!



Spherical angles relate to east longitude (ε) and north latitude (η):



$\theta = \varepsilon$

$\phi = \frac{\pi}{2} - \eta$

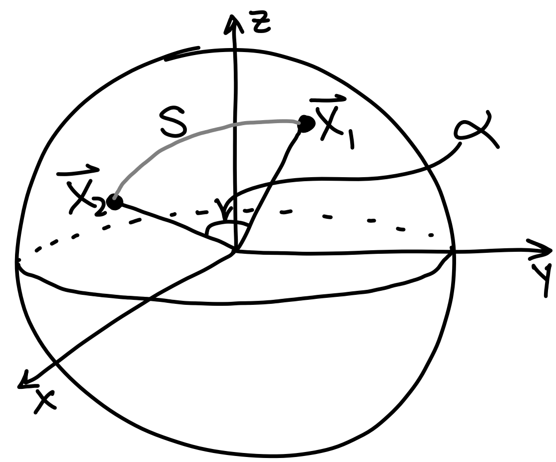
ρ = radius of earth (~3960 mi.)

Given two locations on earth, what is the distance s?

$\varepsilon, \eta \rightarrow \rho, \phi, \theta \rightarrow x, y, z$

$$\vec{x}_1 \cdot \vec{x}_2 = R^2 \cos\alpha$$

$$S = R\alpha$$



Again, can use coordinate relationships to convert equations.

Ex:) $(x-1)^2 + y^2 + z^2 = 1$

$$x^2 + y^2 + z^2 - 2x + 1 = 1$$

$$\rho^2 - 2(\rho \sin\phi \cos\theta) = 0 \quad \Longrightarrow \quad \rho = 2\sin\phi\cos\theta$$

Standard basis vectors

In rectangular coordinates we have

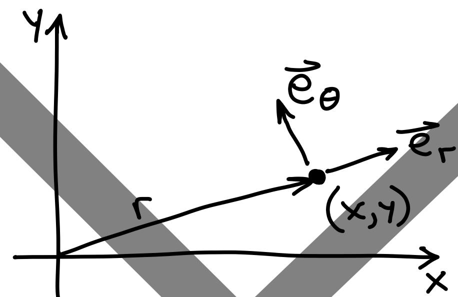
$$\vec{e}_1 = \vec{\imath} = (1,0,0)$$
$$\vec{e}_2 = \vec{\jmath} = (0,1,0)$$
$$\vec{e}_3 = \vec{k} = (0,0,1)$$

Their most important feature (linear combinations!) does not translate well to other coordinate systems.

Also though, they are unit vectors pointing in the direction of increasing a single coordinate.

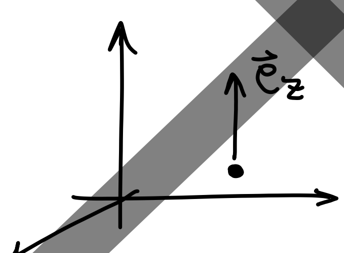
Similarly then:

Polar:



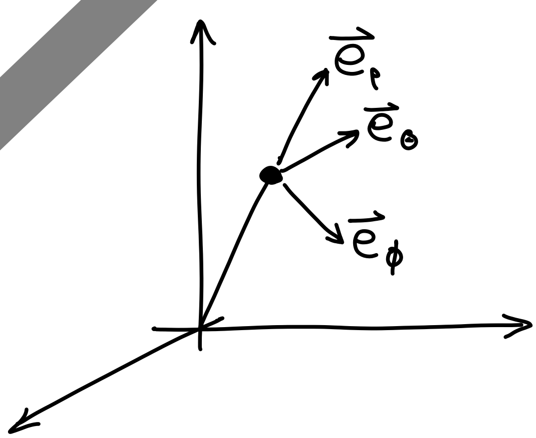
$$\vec{e}_r = \frac{(x,y)}{r}$$
$$\vec{e}_\theta = \frac{(-y,x)}{r}$$

Cylindrical:



$$\vec{e}_z = \vec{e}_3$$

Spherical:



$$\vec{e}_\rho = \frac{(x,y,z)}{\rho}$$
$$\vec{e}_\theta = \frac{(-y,x,0)}{r}$$
$$\vec{e}_\phi = \vec{e}_\theta \times \vec{e}_\rho$$

NB:

- They are not constants!
- They don't work with linear combinations like the rectangular ones do. E.g.:

$\vec{x}$ has rectangular coords x, y, z $\iff$ $\vec{x} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$

but

$\vec{x}$ has spherical coords ρ, ϕ, θ $\not\iff$ $\vec{x} = \rho\vec{e}_\rho + \phi\vec{e}_\phi + \theta\vec{e}_\theta$

- They happen to be orthogonal for each of these coordinate systems.