The Limit Comparison Theorem for Improper Integrals

Limit Comparison Theorem (Type I): If f and g are continuous, positive functions for all values of x, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$$

Then:

1. if $0 < k < \infty$, then

$$\int_a^\infty g(x)\,dx \quad \text{converges} \quad \iff \int_a^\infty f(x)\,dx \quad \text{converges}$$

2. if $k = 0$, then

$$\int_a^\infty g(x)\,dx \quad \text{converges} \quad \implies \int_a^\infty f(x)\,dx \quad \text{converges}$$

3. if $k = \infty$, then

$$\int_a^\infty g(x)\,dx \quad \text{converges} \quad \impliedby \int_a^\infty f(x)\,dx \quad \text{converges}$$

All of the corresponding statements for improper integrals of type II are also true:

Limit Comparison Theorem (Type II): If f and g are continuous, positive functions for all values of x, and

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = k$$

Then:

1. if $0 < k < \infty$, then

$$\int_0^a g(x)\,dx \quad \text{converges} \quad \iff \int_0^a f(x)\,dx \quad \text{converges}$$

2. if $k = 0$, then

$$\int_0^a g(x)\,dx \quad \text{converges} \quad \implies \int_0^a f(x)\,dx \quad \text{converges}$$

3. if $k = \infty$, then

$$\int_0^a g(x)\,dx \quad \text{converges} \quad \impliedby \int_0^a f(x)\,dx \quad \text{converges}$$

These theorems offer an alternative to using the Comparison Theorem (CT) discussed in the book when trying to determine whether an improper integral converges or diverges. The proofs of these three statements use CT, so we can conclude that in some sense, any problem the Limit Comparison Theorem (LCT) can solve could also be solved by CT, just by following the arguments in those proofs; however, sometimes the solution is easier using LCT.

(It should be noted however that there do exist some examples of convergence questions where LCT fails, but CT does not! So strictly speaking, CT is more powerful than LCT.)

First, let's prove part (1.); the proofs of parts (2.) and (3.) are similar (you might want to try working those out for yourself!).

Proof of LCT(I-1.):

We assume that k exists and is a positive finite number, and that the limit from a to ∞ of g converges; we will show that the limit from a to ∞ of f converges. Proving the other direction can be done similarly, or simply by observing that if $\lim \frac{f}{g} = k$ exists and is positive, then $\lim \frac{g}{f} = \frac{1}{k}$ must also exist and be positive...

The definition of the limit tells us that there exists some N such that

$$(k-1) < \frac{f(x)}{g(x)} < (k+1) \quad \text{whenever } x > N.$$

So, for those values of x, we have

$$\frac{f(x)}{g(x)} < (k+1) \quad \Longrightarrow \quad f(x) < (k+1)g(x)$$

We now break the integral in question into two pieces:

$$\int_a^\infty f(x)\,dx = \int_a^N f(x)\,dx + \int_N^\infty f(x)\,dx$$

The first integral is of a continuous function on a closed, bounded interval, so we know that is finite. The convergence of the second integral is concluded by the following, which we can do because of the inequality determined above:

$$\int_N^\infty f(x)\,dx \quad < \quad \int_N^\infty (k+1)g(x)\,dx \quad = \quad (k+1)\int_N^\infty g(x)\,dx$$

(the last integral in the equation above is given to converge; therefore, by the Comparison Theorem, the integral on the left converges.)

We conclude, as desired, that the integral of f converges.

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Example: Determine whether or not the following integral converges or diverges.

$$\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$$

Solution: We feel like the "dominant part" of the integrand is the x in the denominator, and thus expect that this integral should have the same convergence property as

$$\int_1^\infty \frac{1}{x}\,dx$$

which we know diverges. However, we see right away that the integrand in question is actually less than this comparison function, and so CT does not apply.

Of course we saw in class that this comparison function can be modified in such a way that CT will work. However, it is much simpler to use LCT in this case:

$$\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} \lim_{x\to\infty} 1 - e^{-x} = 1$$

This limit is positive and finite, so therefore the two integrals being compared have the same convergence properties. Since one diverges, we immediately conclude that the other (the one we are interested in) must diverge also.