Solutions to the Additional Problems

1. Use the Limit Comparison Theorem to determine whether the integral below converges or diverges:

\[ \int_{0}^{\infty} \frac{1 - e^{-x}}{\sqrt{x + 1}} \, dx \]

Solution: As \( x \) approaches \( \infty \), the dominant part of the integrand is the root in the denominator. So we feel that this integral probably behaves similarly to

\[ \int_{0}^{\infty} \frac{1}{\sqrt{x + 1}} \, dx \]

To be able to make this comparison with LCT, we need to compute

\[ \lim_{x \to \infty} \frac{1 - e^{-x}}{\sqrt{x + 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{x + 1}} = 1 \]

Since this limit is positive and finite, and since both functions are positive and continuous, LCT then tells us that the given integral does have the same convergence property as the comparison integral. So, it diverges.

(over)
2. Use the Limit Comparison Theorem to determine whether the integral below converges or diverges:

\[ \int_0^\infty \frac{(\ln x)(e^{-x})}{(x+1)^2} \, dx \]

**Solution:** This integral has two problems – one at infinity, and one at zero. So we must break it up as

\[ \int_0^\infty \frac{(\ln x)(e^{-x})}{(x+1)^2} \, dx = \int_0^1 \frac{(\ln x)(e^{-x})}{(x+1)^2} \, dx + \int_1^\infty \frac{(\ln x)(e^{-x})}{(x+1)^2} \, dx \]

and determine the convergence or divergence of each of the two pieces.

For the first, the problem is as we approach 0 from the right; we note that as \( x \) approaches zero from the right, it is \( (\ln x) \) that is the dominant part of the integrand. So, we expect that this integral will behave similarly to

\[
\int_0^1 (\ln x) \, dx = \lim_{t \to 0^+} \int_t^1 (\ln x) \, dx \\
= \lim_{t \to 0^+} (x \ln x - x)_t^1 \\
= \lim_{t \to 0^+} (-1) - (t \ln t - t) \\
= (-1) - \lim_{t \to 0^+} t \ln t + \lim_{t \to 0^+} t \\
= (-1) - \lim_{t \to 0^+} \frac{\ln t}{t} + 0 \\
= (-1) - \lim_{t \to 0^+} \frac{1}{t} \\
= (-1) - 0 = -1
\]

which we see converges.

In this case of course, our integrand is NOT positive, since \( \ln x \) is \( \leq 0 \) for all values of \( x \) in the domain of the integral. This can be fixed easily however by observing that we can instead consider

\[ \int_0^\infty -\frac{(\ln x)(e^{-x})}{(x+1)^2} \, dx = -\int_0^\infty \frac{(\ln x)(e^{-x})}{(x+1)^2} \, dx \]

which is positive, and of course has the same convergence property as the original integral; we do the same for the comparison function.

\[ \int_0^1 -(\ln x) \, dx = -\int_0^1 (\ln x) \, dx \]

(over)
To be able to make this comparison with LCT, we need to compute

\[
\lim_{x \to 0^+} \frac{-(\ln x)(e^{-x})}{(x+1)^2} = \lim_{x \to 0^+} \frac{(e^{-x})}{(x+1)^2} = 1
\]

Since this limit is positive and finite, and since both functions are positive and continuous, LCT then tells us that the given integral does have the same convergence property as the comparison integral. So, it converges.

Now we consider the second integral. In this integral the problem is as \(x\) approaches infinity; and in that limit, it is the term \((e^{-x})\) that is the dominant part of the integrand. So, we expect that this integral will behave similarly to

\[
\int_1^\infty (e^{-x}) \, dx = \lim_{t \to \infty} \int_1^t (e^{-x}) \, dx = \lim_{t \to \infty} (e^{-x} \big|_1^t) = \lim_{t \to \infty} (e^{-1} - e^{-t}) = e^{-1}
\]

which we see converges.

To be able to make this comparison with LCT, we need to compute

\[
\lim_{x \to \infty} \frac{\ln x (e^{-x})}{(e^{-x})(x+1)^2} = \lim_{x \to \infty} \frac{\ln x}{(x + 1)^2} = \lim_{x \to \infty} \frac{1}{2(x + 1)} = 0
\]

Since this limit is zero, and since both functions are positive and continuous, the LCT tells us that the convergence of the integral of \(e^{-x}\) implies the convergence of the original integral. Of course the integral of \(e^{-x}\) does converge, so, this integral converges too.

The original integral is thus the sum of two convergent improper integrals, so we conclude that it does converge.