EXAM I
Math 51, Spring 2002.
You have 2 hours.
No notes, no books, no calculators.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT
Good luck!

Name ________________________________
ID number __________________________

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Signature: __________________________

Circle your TA’s name:

Tarn Adams (2 and 6)
Mariel Saez (3 and 7)
Yevgeniy Kovchegov (4 and 8)
Heaseung Kwon (A02)
Alex Meadows (A03)

Total ___________ (/100 points)  
Circle your section meeting time:

11:00am  1:15pm  7pm
1. Let

\[ \vec{u} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \]

(a) Compute the length of each of the given vectors, and the dot product for each pair.

(b) Let \( \theta_{\vec{u},\vec{v}} \) be the angle formed at the origin between the vectors \( \vec{u} \) and \( \vec{v} \). Evaluate \( \cos(\theta_{\vec{u},\vec{v}}) \)
(c) Find a linear dependence of the three vectors given, or prove that they are independent.
2. (a) Use pivots to prove that a collection of \((n + 1)\) vectors in \(\mathbb{R}^n\) must have a linear dependence. (Make sure to explain ALL of your reasoning as carefully and clearly as possible.)
(b) Prove that if the system of equations represented by

\[ A \vec{x} = \vec{b} \]

where

\[ A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    a_{m1} & \cdots & & a_{mn}
\end{pmatrix} \]

has a solution, then we can conclude that \( \vec{b} \) is in the column space of \( A \).
3. (a) Find the null space for the matrix below.

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & -1
\end{pmatrix}
\]

(b) Use your result from part (a) to find a parametric representation of the solution set to the system of equations below WITHOUT row reducing; explain how you know your answer is complete.

\[
x + y = 1 \\
y + z = 0 \\
x - z = 1
\]
4. Use the Cauchy-Schwarz inequality

\[ |\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \|\vec{w}\| \]

to prove the Triangle Inequality:

\[ \|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \]

(Hint: Begin by computing \(\|\vec{v} + \vec{w}\|^2\) with dot products, and then plug in the Cauchy-Schwarz inequality when the opportunity arises.)
5. The matrix $A$ below has the given reduced row echelon form (You do not need to verify this).

$$A = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 3 & 2 & 3 \\ 3 & 2 & 2 & 6 \\ 4 & 1 & 2 & 7 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using this information, write down bases for the null space, column space, and row space of $A$. 


**Bonus Question:** Let the function $f$ project vectors in $\mathbb{R}^3$ to the $xy$-plane; in other words,

$$f \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Also suppose that the three vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

are all perpendicular to each other.

Show that it is NOT possible for all three of the angles created at the origin by the vectors $f(\vec{u})$, $f(\vec{v})$, $f(\vec{w})$ to be acute.

(Hint: Try phrasing the given conditions on the three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ in terms of dot products, and then expand those expressions in terms of components.)