C13S02.003: If either \( x \) or \( y \) is nonzero, then \( x^2 + y^2 > 0 \), and so \( f(x, y) \) is defined—but not if \( x = y = 0 \). Hence the domain of \( f \) consists of all points \((x, y)\) in the plane other than the origin.

C13S02.005: The real number \( z \) has a unique cube root \( z^{1/3} \) regardless of the value of \( z \). Hence the domain of \( f(x, y) = (y - x^2)^{1/3} \) consists of all points in the \( xy \)-plane.

C13S02.006: The real number \( z \) has a unique cube root \( z^{1/3} \) regardless of the value of \( z \). But \( \sqrt{2x} \) is real if and only if \( x \geq 0 \). Therefore the domain of \( f(x, y) = (2x)^{1/2} + (3y)^{1/3} \) consists of all those points \((x, y)\) for which \( x \geq 0 \).

C13S02.033: The level curves of \( f(x, y) = 4x^2 + y^2 \) are ellipses centered at the origin, with major axes on the \( x \)-axis and minor axes on the \( y \)-axis.

C13S02.052: The graph of \( f(x, y) = 2y^3 - 3y^2 - 12y + x^2 \) is shown in Fig. 13.2.29. The fact that the derivative of \( g(y) = 2y^3 - 3y^2 - 12y \) is zero when \( y = -1 \) and when \( y = 2 \) accounts for the “waviness” of the figure in the \( y \)-direction.

Section 13.3

C13S03.006: The sum, product, and quotient of continuous functions is continuous where defined, so

\[
f(x, y) = \frac{9 - x^2}{1 + xy}
\]

is continuous where \( xy \neq -1 \). Consequently

\[
\lim_{(x,y) \to (2,3)} f(x, y) = \frac{9 - 2^2}{1 + 2 \cdot 3} = \frac{5}{7}.
\]
Given:

\[ f(x, y) = \frac{2x^2y}{x^4 + y^2}. \]

Suppose that \((x, y)\) approaches \((0, 0)\) along the nonvertical straight line with equation \(y = mx\). If \(m \neq 0\), then—on that line—

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{2mx^3}{x^4 + m^2x^2} = \lim_{x \to 0} \frac{2mx}{x^2 + m^2} = 0
\]

Clearly if \(m = 0\) the result is the same. And if \((x, y)\) approaches \((0, 0)\) along the y-axis, then—on that line—

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{y \to 0} \frac{2 \cdot 0 \cdot y}{0 + y^2} = 0.
\]

Therefore as \((x, y)\) approaches \((0, 0)\) along any straight line, the limit of \(f(x, y)\) is zero. But on the curve \(y = x^2\) we have

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{2x^4}{x^4 + x^4} = 1,
\]

and therefore the limit of \(f(x, y)\) does not exist at \((0, 0)\) (see the Remark following Example 9). For related paradoxical results involving functions of two variables, see Problem 60 of Section 13.5 and the miscellaneous problems of Chapter 13.

Given:

\[ f(x, y) = \frac{xy}{x^2 + y^2}. \]

After we convert to polar coordinates, we have

\[ f(r, \theta) = \frac{r^2 \cos \theta \sin \theta}{r^2} = \cos \theta \sin \theta = \frac{1}{2} \sin 2\theta. \]

On the hyperbolic spiral \(r \theta = 1\), we have \(\theta \to +\infty\) as \(r\) approaches zero through positive values. Hence \(f(r, \theta)\) takes on all values between \(-\frac{1}{2}\) and \(\frac{1}{2}\) infinitely often as \(r \to 0^+\). Therefore, as we discovered in Example 9, \(f(x, y)\) has no limit as \((x, y) \to (0, 0)\).

Section 13.4

If \(f(x, y) = e^x(\cos y - \sin y)\), then

\[
\frac{\partial f}{\partial x} = e^x(\cos y - \sin y) \quad \text{and} \quad \frac{\partial f}{\partial y} = -e^x(\cos y + \sin y).
\]

If \(f(r, s) = \frac{r^2 - s^2}{r^2 + s^2}\), then

\[
\frac{\partial f}{\partial r} = \frac{4rs^2}{(r^2 + s^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial s} = -\frac{4r^2s}{(r^2 + s^2)^2}.
\]
If \( z(x, y) = xy \exp(-xy) \), then
\[
\begin{align*}
  z_x(x, y) &= y \exp(-xy) - xy^2 \exp(-xy), \\
  z_y(x, y) &= x \exp(-xy) - x^2 y \exp(-xy), \\
  z_{xy}(x, y) &= -xy \exp(-xy) - (xy - 1) \exp(-xy) + xy(xy - 1) \exp(-xy) \\
                   &= (x^2 y^2 - 3xy + 1) \exp(-xy) = z_{yx}(x, y).
\end{align*}
\]

The graph of the given equation \( z = 3x + 4y \) is a plane, so it is its own tangent plane and the coordinates of the point of tangency don't matter. Answer: An equation of the plane tangent to the graph of \( z = 3x + 4y \) at the point \( P \) is \( z = 3x + 4y \).

Given \( f(x, y) = xy \) and the point \( P(1, -1, -1) \) on its graph. Then \( f_x(x, y) = y \) and \( f_y(x, y) = x \), so \( f_x(1, -1) = -1 \) and \( f_y(1, -1) = 1 \). By Eq. (11) an equation of the plane tangent to the graph of \( z = f(x, y) \) at the point \( P \) is \( z + 1 = -(x - 1) + (y - 1) \); that is, \( x - y + z = 1 \).

If \( f_x(x, y) = \cos^2 xy \) and \( f_y(x, y) = \sin^2 xy \), then
\[
f_{xy}(x, y) = -2x \sin xy \cos xy \neq -2y \sin xy \cos xy = f_{yx}(x, y).
\]
By the Note preceding and following Eq. (14), there can be no function \( f(x, y) \) having the given first-order partial derivatives.

Given \( f_x(x, y) = \cos x \sin y \) and \( f_y(x, y) = \sin x \cos y \), we find that
\[
f_{xy}(x, y) = \cos x \cos y = f_{yx}(x, y).
\]
So it's not impossible that there exists a function \( f(x, y) \) having the given first-order partial derivatives. Indeed, by inspection, one such function is \( f(x, y) = \sin x \sin y \).

If \( u(x, t) = \exp(-n^2 kt) \sin nx \) where \( n \) and \( k \) are constants, then
\[
\begin{align*}
  u_t(x, t) &= -n^2 k \exp(-n^2 kt) \sin nx, \\
  u_x(x, t) &= n \exp(-n^2 kt) \cos nx, \\
  \text{and} \quad u_{xx}(x, t) &= -n^2 \exp(-n^2 kt) \sin nx.
\end{align*}
\]
Therefore \( u_t = ku_{xx} \) for any choice of the constants \( k \) and \( n \).
Part (a): If \( y(x, t) = \sin(x + at) \) (where \( a \) is a constant), then

\[
\begin{align*}
y_t(x, t) &= a \cos(x + at), \\
y_{tt}(x, t) &= -a^2 \sin(x + at), \\
y_z(x, t) &= \cos(x + at), \\
y_{zz}(x, t) &= -\sin(x + at).
\end{align*}
\]

Therefore \( y_{tt} = a^2 y_{zz} \).

Part (b): If \( y(x, t) = \cosh(3(x - at)) \), then

\[
\begin{align*}
y_t(x, t) &= -3a \sinh(3(x - at)), \\
y_{tt}(x, t) &= 9a^2 \cosh(3(x - at)), \\
y_z(x, t) &= 3 \sinh(3(x - at)), \\
y_{zz}(x, t) &= 9 \cosh(3(x - at)).
\end{align*}
\]

Therefore \( y_{tt} = a^2 y_{zz} \).

Part (c): If \( y(x, t) = \sin kx \cos kat \) (where \( k \) is a constant), then

\[
\begin{align*}
y_t(x, t) &= -ka \sin kx \sin kat, \\
y_{tt}(x, t) &= -k^2 a^2 \sin kx \cos kat, \\
y_z(x, t) &= k \cos kx \cos kat, \\
y_{zz}(x, t) &= -k^2 \sin kx \cos kat.
\end{align*}
\]

Therefore \( y_{tt} = a^2 y_{zz} \).

C13S04.060: Given: The constant \( q \) and the function

\[
\phi(x, y, z) = \frac{q}{\sqrt{x^2 + y^2 + z^2}}.
\]

Then

\[
\begin{align*}
\phi_x(x, y, z) &= -\frac{qx}{(x^2 + y^2 + z^2)^{3/2}} \\
\phi_{xx}(x, y, z) &= \frac{q(2x^2 - y^2 - z^2)}{(x^2 + y^2 + z^2)^{5/2}}.
\end{align*}
\]

By the symmetries in \( \phi \) among \( x, y, \) and \( z \), it follows that

\[
\phi_{yy}(x, y, z) = \frac{q(2y^2 - x^2 - z^2)}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{and} \quad \phi_{zz}(x, y, z) = \frac{q(2z^2 - x^2 - y^2)}{(x^2 + y^2 + z^2)^{5/2}}.
\]

It is now clear that \( \phi \) satisfies the three-dimensional Laplace equation \( \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \).

C13S04.065: If \( f(x, y) = x^2 + 2xy + 2y^2 - 6x + 8y \), then the equations

\[
\begin{align*}
f_x(x, y) &= 0, \\
f_y(x, y) &= 0
\end{align*}
\]

are

\[
\begin{align*}
2x + 2y - 6 &= 0, \\
2x + 4y &= -8.
\end{align*}
\]

which have the unique solution \( x = 10, \ y = -7 \). So the surface that is the graph of \( z = f(x, y) \) contains exactly one point at which the tangent plane is horizontal, and that point is \( (10, -7, -58) \).
C13S04.068: If \( z(x, y) = \ln(\cos x) - \ln(\cos y) \), then

\[
\begin{align*}
z_x(x, y) &= -\tan x, & z_y(x, y) &= \tan y, \\
z_{xx}(x, y) &= -\sec^2 x, & z_{xy}(x, y) &\equiv 0, \quad \text{and} \\
z_{yy}(x, y) &= \sec^2 y.
\end{align*}
\]

Therefore

\[
(1 + z_y^2)z_{xx} - z_{x}z_{xy}z_{xy} + (1 + z_x^2)z_{yy} = -\sec^2 x \sec^2 y - 0 + \sec^2 x \sec^2 y \equiv 0.
\]

C13S04.070: The sum of two harmonic functions is harmonic. Here is a proof for the case of functions of two variables. Suppose that \( f \) and \( g \) are harmonic. Then

\[
f_{xx} + f_{yy} = 0 \quad \text{and} \quad g_{xx} + g_{yy} = 0.
\]

Let \( h(x, y) = f(x, y) + g(x, y) \). Then

\[
h_{xx} + h_{yy} = f_{xx} + g_{xx} + f_{yy} + g_{yy} = f_{xx} + f_{yy} + g_{xx} + g_{yy} = 0 + 0 = 0.
\]

Therefore \( f + g \) is harmonic. This concludes the proof. By induction, you may extend it to show that the sum of any finite number of harmonic functions is harmonic. This answers the question in Problem 70.