Present Value and Continuous Money Streams

Introduction

The “value of money” is a very difficult topic to discuss in full detail, because there are so many parameters to consider, and so many different points of view.

In this class, we will take a simplified perspective which, while clearly being imperfect, will allow for what I feel is a very intuitive and straight-forward approach for solving a very large class of financial questions.

In particular, for the purposes of this class, we will assume in any given computation that there is a single number, which we will call the “interest rate”, which will be used to determine the way in which the intrinsic value of money changes over time. In real life, this is not the case, since interest rates depend on the type of issue being discussed, may or may not be constant over time, and of course the whole problem is further complicated by the existence of inflation. Again though, this extremely simplified view will turn out to be sufficient for our purposes in this class.

Setup

The “interest rate” \((r)\) is the instantaneous rate at which a sum of money accumulates, expressed as a fraction of the sum. For example, suppose you have a bank account; the bank will add dollars into your account over time, as described by the equation

\[
\frac{dB(t)}{dt} = r B(t)
\]

where \(B(t)\) is the balance in your account at time \(t\). As suggested above, this equation says that \(r\) is the rate at which the balance increases, divided by the balance itself.

This equation is called a “differential equation”; we will study these later in this course. At that time, we will see that from this equation we can solve directly for \(B(t)\), which turns out to be

\[
B(t) = B_0 e^{rt}
\]

where \(B_0\) is the initial balance in your account at time \(t = 0\).

The point of view we will take in this course is that the bank increases the number of dollars in your account to acknowledge that the individual dollars in that account are becoming less valuable over time. Specifically, they increase the number of dollars in your account in such a way that the sum retains the same intrinsic value at all times.
The equation can then be interpreted to say that the factor $e^{rt}$ is the ratio between the number of dollars at time $t$ and the number of dollars at time $0$ that yield the same intrinsic value. Turning this around, we could further interpret the factor $e^{rt}$ as the ratio between the intrinsic value of a single dollar at time $t$, and the intrinsic value of a single dollar at time $0$.

Despite the fact that the intrinsic value of a dollar changes over time, the standard convention is to still use the single unit “dollar” to refer to it at all times. Of course this is necessary since we also use the word “dollar” to refer to the actual note we use as currency.

To avoid confusion between the use of “dollar” to refer to the currency note and to the intrinsic value of that note, in this course we will use a slightly different notation to refer to the intrinsic value of a dollar being discussed. This notation will acknowledge that the value of a dollar varies with time, by including the time in question in the notation. Specifically, we will define

$$S_t = \text{the intrinsic value of one dollar, at time } t$$

This notation is not at all standard, and will probably not be seen outside of this course. Furthermore, as suggested in the introduction, this notation has serious flaws that do not allow for it to be used in more rigorous discussions of these issues. It will however be perfectly sufficient for the problems we will discuss in this course.

In any case where we have different units that measure the same type of quantity, we have a unit conversion formula that tells us how to convert between these different units. For example, the equation $(\text{ft}) = 12(\text{in})$ is the unit conversion formula that allows us to convert a length expressed in feet to a length expressed in inches. For example,

$$3(\text{ft}) = 3 \left( 12(\text{in}) \right) = 36(\text{in})$$

Of course the expressions “$3 \text{ ft}$” and “$36 \text{ in}$” represent the same actual length – just with respect to different units of length. So using the unit conversion formula does not actually change the value of an expression; it just rephrases it in terms of different units.

Similarly, we have a unit conversion formula for the units $S_t$, for different times $t$. Specifically, we have

$$(S_a) = e^{r(b-a)}(S_b)$$

This can be derived from the previous formulas for an account balance, by equating the intrinsic values of the account at times $a$ and $b$, and solving for $(S_a)$:

$$B_0 e^{ra}(S_a) = B_0 e^{rb}(S_b)$$

$$(S_a) = e^{b-a}(S_b)$$

**Ex. 1:** Using an interest rate of 5%/yr, how many dollars in the year 2042 will it take to be equivalent to 13 dollars in the year 2003?
**Sol.:** There are two sums of money that we want to insist have the same intrinsic value, with respect to the given interest rate. We will express this with the equation

\[ 13 \text{($2003$)} = x \text{($2042$)} \]

since the left side expresses the intrinsic value of 13 dollars in the year 2003, and the right side expresses the intrinsic value of some unknown number (the number we are looking for) of dollars in the year 2042.

To solve for \( x \) we must rewrite one of the units in terms of the other, so that we can cancel the units. It does not matter which we choose to rewrite. Arbitrarily, let’s rewrite the units on the left side. In this case, the unit conversion formula becomes

\[ \text{($2003$)} = e^{(0.05/\text{yr})(2012 \text{ yr} - 2003 \text{ yr})} \text{($2042$)} = e^{30/20} \text{($2042$)} \]

Plugging this into our original equation, we can then cancel the units, and solve for \( x \):

\[
\begin{align*}
13 \text{($2003$)} &= x \text{($2042$)} \\
13e^{30/20} \text{($2042$)} &= x \text{($2042$)} \\
13e^{30/20} &= x
\end{align*}
\]

**Definition:** The “present value” of a quantity of money is the number of today’s dollars that has the same intrinsic value as the given quantity.

**Ex. 2:** In exchange for some work you do on his computer, a friend signs a contract promising to pay you $500, but not until exactly one year from today. Assuming an interest rate of 10%/yr, what is the present value of that contract?

**Sol.:** Again we equate the intrinsic values of two quantities. In this case, we will suppress the units of time, since they will clearly cancel; also, we will arbitrarily define today to be the time \( t = 0 \):

\[ 500(1) = x(0) \]

Again we use the unit conversion formula, which in this case is:

\[ (1) = e^{(1)(0-1)}(0) = e^{-1}(0) \]

Plugging this into our original equation, we get

\[
\begin{align*}
500(1) &= x(0) \\
500e^{-1}(0) &= x(0) \\
500e^{-1} &= x
\end{align*}
\]

So, the present value of the contract is \( 500e^{-1} \approx 452.42 \) dollars.
Continuous Money Streams

Very often in the real world, debts are repaid with regular monthly payments over an extended period of time. For example, when you purchase a car, you might very well agree to pay $500 per month for five years, rather than pay the full price of the car at the time of purchase (since you very likely will not have the full amount at the time of purchase).

Of course when determining your monthly payment, the dealer will not agree to simply divide the price of the car by 60 – because, of course, the money that you are repaying him with are future dollars and thus are less valuable than the current dollars that are being used to express the price of the car.

Further complicating the issue is the fact that the dollars you will be using to pay for the car do not themselves even all have the same value – the ones you will be making your first few payments with are almost as valuable as current dollars, while the ones you will make your final few payments with are of course worth much less.

Determining the correct monthly payment involves writing down the present value of each of the monthly payments, adding them up, and then insisting that sum equals the price of the car. This turns out to be algebraically inconvenient.

However, we can come up with a very close approximation to the right answer by changing the problem slightly. Namely, instead of making discrete payments at the beginning of each month, let’s assume instead that the monthly payments are spread out continuously over the entire month, forming a “continuous payment stream”.

Of course, a truly continuous payment stream cannot actually be performed physically, because all of the units of legal tender (pennies, dimes, dollar bills,...) are discrete quantities of money.

To help us visualize a continuous payment stream, let’s consider the following slightly preposterous idea, which would of course never actually happen, even though there is no reason why it couldn’t. Suppose that a new type of currency comes out, which is a liquid. Arbitrarily, let us suppose that one liter of this liquid is defined to have the same value as one dollar. So, a monthly payment of say $500 could be paid discretely by giving the car dealer 500L of this liquid money at the beginning of each month.

The corresponding continuous payment stream would simply be to connect a hose from the container in which you keep your liquid money to the container in which he keeps his liquid money, and allowing the liquid money to flow at a rate such that over the course of each month, a total of 500L is transferred.

The continuous payment stream of course is not precisely the same as the discrete monthly payments, but it can be shown without too much effort that the present value of a continuous payment stream is very close to that of the corresponding discrete payments. And, continuous payment streams have the very nice property that their intrinsic values can be represented very conveniently with integrals, as we will see below.
Value of Continuous Money Streams

A continuous money stream can be represented with a graph of the pay rate as a function of time. For example, a constant payment rate of $500 per month ($6000 per year) over five years could be represented by

To determine the present value of such a stream, we think of this continuous payment stream as being the sum of “slices” of money, just like we have viewed areas and volumes and such as being sums of slices.

In this case, the slice of money at time $t$ is just the money that is paid between the times $t$ and $t + dt$.

As usual, there is an underlying Riemann sum that we denote for convenience with integral notation, and then eventually turn into an actual integral. In this case, we have

$$\text{Value}(V) = \int dV = \int (\text{pay rate}) \, dt$$

Of course the pay rate referred to in the last expression above is in terms of the dollars at the time that slice of money happens. So, in this case, we get

$$\int (\text{pay rate}) \, dt = \int_{0}^{5} 6000(\$) \, dt$$

This integral does represent the value of the given payment stream. However, we cannot evaluate this integral in its current form, since the units ($\$_t$) are not constant, and thus cannot be factored outside of the integral. In order to evaluate the integral, we must first rewrite the nonconstant unit ($\$_t$) in terms of some constant units. Since in this case we are interested in
determining the present value, we will convert into the units ($0), again using the same unit conversion formula. This gives us

\[ \int_0^5 6000(t)\, dt = \int_0^5 6000e^{(0-t)}(0)\, dt = \int_0^5 6000e^{-rt}\, dt(0) \]

This integral can now be evaluated, and gives us the present value of that payment stream since the units are ($0). Using an interest rate of 10%/yr, this works out to be about $23,608.

**Ex. 3:** We could interpret the result above as saying that if you wanted to purchase a car worth $23,608, and if the dealer agreed to give you an annual interest rate of 10% over five years, then your monthly payment would be $500.

Note that over the course of those five years, you actually pay the dealer 30,000 dollars, in some sense...namely, you give the dealer 30,000 actual dollar bills. But this is necessary since the dollars you give the dealer are worth less than the present dollars that are being used to describe the value of the car.

Given this reasoning, what do you think would happen to the value of the car that you could afford to purchase if the interest rate the dealer gave you were higher, say 12%? First try to use intuition about the value of the dollars you are using in your payments, and then work this problem out for yourself to check that your intuition was correct.

**Ex. 4:** Suppose that you make continuous payments into a bank account at a rate of $500 per month, over the course of five years. Suppose also that this bank account earns an annual interest rate 10%. How much money will be in the account at the end of those five years?

**Sol.:** Although this appears at first glance to be a completely different question, this is in fact almost exactly the same question as in the previous example! In particular, just like in the previous example, this question is asking for the value of a continuous payment stream, with the same pay rate, duration, and interest rate. So, the integral that we begin with is the same.

This time however, we convert the units into ($5) instead of ($0), since the time we are interested in is $t = 5$:

\[ \int_0^5 6000(t)\, dt = \int_0^5 6000e^{(5-t)}(5)\, dt = e^{5r} \int_0^5 6000e^{-rt}\, dt(5) \]

This integral can now be evaluated, and gives us the value of that payment stream at $t = 5$ since the units are ($5). Using an interest rate of 10%/yr, this works out to be about $38,923.

But there is an even easier way to do this problem. The only difference between this problem and the previous is that it is asking for the value of that money stream to be expressed in terms of different units – specifically, in terms of the units $5$, since it is asking for the number of dollars in the account at time $t = 5$.

So, all we need to do is to take the previously computed value (expressed in terms of present dollars), and convert it into future dollars using the unit conversion formula. This gives us

\[ 23,608(0) = 23,608e^{(.1)(5-0)}(5) \approx 38,923(5) \]
So, we again conclude that there would be 38,923 dollars in the account.

**Ex. 5:** Suppose you decide that you want to purchase a car which is more expensive than the car from example 3. In particular, you decide to buy a convertible Corvette for $50,000. Assuming that you want to pay it off over 5 years with an interest rate of 10%/yr, what would your monthly payment be?

**Sol.:** This problem is again a present value computation; but instead of using the payment rate to determine the present value, we will use the present value to determine the payment rate.

Call the payment rate \( P \). The present value of the money stream of payments is thus

\[
V = \int dV = \int_0^5 P S_t \, dt
\]

Since this money stream is intended to pay off a car worth 50,000(\( S_0 \)), we equate those two amounts; and then to solve for \( P \), we must convert the nonconstant units in the integral into present dollars and then evaluate the integral:

\[
50,000(\$_0) = \int_0^5 P S_t \, dt = \int_0^5 Pe^{r(0-t)} S_0 \, dt = \int_0^5 e^{-rt} \, dt \cdot P(\$_0)
\]

Cancelling units, evaluating the integral, and solving for \( P \), we conclude that \( P \approx 12,707 \). This is your annual payment rate, corresponding to a monthly payment rate of about $1059.

**Ex. 6:** Suppose that you decide you cannot afford to pay that much per month. So instead, you decide to try to extend your payments over a longer period of time, in the hopes that you can bring your monthly payment down to $800. How many years would this require for you to continue to make payments?

**Sol.:** We want to determine the time \( T \) such that the value of the stream of payments of $800 per month (= $9600 per year) is the same as the value of the car. We get

\[
50,000(\$_0) = \int_0^T 9600 S_t \, dt = \int_0^T 9600e^{r(0-t)} S_0 \, dt = \int_0^T e^{-rt} \, dt 9600(\$_0)
\]

\[
\frac{125}{24} = \int_0^T e^{-rt} \, dt
\]

\[
= -\frac{1}{r} e^{-t/10} \bigg|_0^T
\]

\[
= 10 \left( 1 - e^{-T/10} \right)
\]

\[
T = -\frac{10 \ln \left( \frac{115}{240} \right)}{10} \approx 7.35 \text{ years}
\]

**Ex. 7:** Noticing that longer time loans give you lower payments, you decide to see just how low you can get your monthly payment to be. Instead of paying off the loan over 5 years or 7.35
years or 20 years, you decide to see just how low your payment would be if you took an infinite amount of time to pay off the loan (this of course would be the minimum possibly payment rate). What is that payment rate?

**Sol.** We solve for the payment in the same way as we did in example 5; this time however, the integral is an improper integral:

\[ V = \int dV = \int_0^\infty P \$t \, dt \]

\[
50,000(\$0) = \lim_{T \to \infty} \int_0^T P \$t \, dt \\
= \lim_{T \to \infty} \int_0^T Pe^{(0-i)\$0} \, dt \\
= \lim_{T \to \infty} \int_0^T e^{-t/10} \, dt P \$0 \\
= \lim_{T \to \infty} \left( -10e^{-t/10} \right)_0^T P \$0 \\
= \lim_{T \to \infty} 10 \left( 1 - e^{-T/10} \right) P \$0 \\
= 10P \$0
\]

So, we conclude that the payment rate would be $5000 per year, which is about $417 per month.

This seems surprising, that even though we make these payments over an infinite amount of time, the value is finite. But we must remember of course that the value of the payments is actually decreasing exponentially over time, since the dollars we are paying with are losing value exponentially over time.

Another point of view on this problem would be to think of it in terms of the rate of change of the account balance of our debt to the car dealer. In this particular case, since we want the loan to be repaid over an infinite amount of time, this means that we want the account balance to remain constant. Of course interest is making a positive contribution to the account balance, at a rate given by the equation

\[
\frac{dB(t)}{dt} = r B(t)
\]

In order for the account balance to actually remain constant, the payments you make must counter this exactly. In other words, your pay rate must be exactly \( rB \). This gives us \((.1)(50,000) = 5,000\) dollars per year, just as we concluded with the previous integral.