2. \( W = \int_0^1 \cos\left(\frac{1}{2} \pi x\right) dx = \left[ -\frac{\sin(\frac{1}{2} \pi x)}{\frac{1}{2} \pi} \right]_0^1 = -\frac{2}{\pi} \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = 0 \) N-m = 0 J.

**Interpretation:** From \( x = 1 \) to \( x = \frac{1}{2} \), the force does work equal to \( J^{3/2} \cos(\frac{1}{2} \pi x) dx = \frac{3}{\pi} \left( 1 - \frac{\sqrt{2}}{2} \right) J \) in accelerating the particle and increasing its kinetic energy. From \( x = \frac{1}{2} \) to \( x = 2 \), the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from \( x = 1 \) to \( x = \frac{1}{2} \).

5. (a) If \( J^{0.12} k_x dx = 2 J \), then \( 2 = \left[ \frac{1}{2} k x^2 \right]_{0}^{0.12} = \frac{1}{2} k (0.0144) = 0.0072 k \) and

\[
\frac{k}{0.0072} = \frac{2800}{9} \approx 277.8. \quad \text{Thus, the work needed to stretch the spring from 35 cm to 40 cm is}
\]

\[
J^{0.10} = \int_{0.05}^{0.20} k x dx = \left[ \frac{1}{2} k x^2 \right]_{0.05}^{0.20} = \frac{1}{2} k \left( \frac{1}{100} - \frac{1}{400} \right) = \frac{25}{84} \approx 1.04 J.
\]

(b) \( f(x) = kx \), so \( 30 = \int_0^{0.20} f(x) dx = \frac{2800}{9} x \) and \( x = \frac{270}{2800} m = 0.98 \) cm

6. Let \( L \) be the natural length of the spring in meters. Then

\[
6 = J^{0.12-L} k x^2 dx = \left[ \frac{1}{2} k x^2 \right]_0^{0.12-L} = \frac{1}{2} k \left( (0.12 - L)^2 - (0.10 - L)^2 \right)
\]

\[
10 = J^{0.14-L} k x^2 dx = \left[ \frac{1}{2} k x^2 \right]_0^{0.14-L} = \frac{1}{2} k \left( (0.14 - L)^2 - (0.12 - L)^2 \right). \quad \text{Simplifying gives us}
\]

\[
12 = k(0.0044 - 0.04L) \quad \text{and} \quad 20 = k(0.0052 - 0.04L). \quad \text{Subtracting the first equation from the second gives}
\]

\[
8 = k(0.0008), \quad k = 10,000. \quad \text{Now the second equation becomes} \quad 20 = 52 - 400L, \quad \text{so} \quad L = \frac{38}{40} \text{ cm}.
\]

9. The work needed to lift the cable is \( W \approx \lim_{n \to \infty} \sum_{i=1}^{n} a_i \Delta x = J_0^{500} 2x dx = \left[ x^2 \right]_0^{500} = 250,000 \text{ ft-lb} \). The work needed to lift the coal is 800 lb · 500 ft = 400,000 ft-lb. Thus, the total work required is 250,000 + 400,000 = 650,000 ft-lb.

10. A "slice" of water \( \Delta x \) thick and lying at a depth of \( x_i \) m (where \( 0 \leq x_i \leq \frac{L}{2} \)) has a volume \( 2 \times 1 \times \Delta x \) m³, a mass of \( 2000 \Delta x \text{ kg} \), weighs about \( 9.8)(2000 \Delta x) = 19,600 \Delta x \text{ N} \), and thus requires about \( 19,600x_i \Delta x \) J of work for its removal. So

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} 19,600x_i \Delta x = J_0^{500} 19,600 x dx = \left[ 9800x^2 \right]_0^{500} = 2450 J
\]

11. A horizontal cylindrical slice of water \( \Delta x \) ft thick has a volume of \( \pi r^2 h = \pi \cdot 12^2 \cdot \Delta x \) ft³ and weighs about \( (62.5 \text{ lb/ft}^3)(144\pi \Delta x \text{ ft}^3) = 9000\pi \Delta x \text{ lb} \). If the slice lies \( x_i \) ft below the edge of the pool (where \( 1 \leq x_i \leq 5 \)), then the work needed to pump it out is about \( 9000\pi x_i \Delta x \). Thus,

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} 9000\pi x_i \Delta x = J_0^{5} 9000\pi x dx = \left[ 45000\pi x^2 \right]_0^{5} = 4500\pi(25 - 1) = 108,000\pi \text{ ft-lb}
\]

12. Let \( x \) be depth in feet, so that \( 0 \leq x \leq 5 \). Then \( \Delta W = (62.5\pi(\sqrt{5^2-x^2})^2 \Delta x \cdot x \text{ ft-lb} \) and

\[
W \approx 62.5\pi J_0^{5} x(25 - x^2) dx = 62.5\pi \left( \frac{25}{4} x^2 - \frac{1}{4} x^4 \right) = 62.5\pi \left( \frac{625}{4} - \frac{625}{4} \right) = 62.5\pi \left( \frac{625}{4} \right)
\]

\[
\approx 3.07 \times 10^4 \text{ ft-lb}
\]

15. \( V = \pi r^2 z \), so \( V \) is a function of \( z \) and \( P \) can also be regarded as a function of \( z \). If \( V_1 = \pi r^2 z_1 \) and \( V_2 = \pi r^2 z_2 \), then

\[
W = \int_{V_1}^{V_2} P(z) dz = \int_{V_1}^{V_2} \pi r^2 P(V(x)) dx = \int_{V_1}^{V_2} P(V(x)) dV(x) \quad \text{[Let} V(x) = \pi r^2 x, \text{so} dV(x) = \pi r^2 dx].}
\]

\[
= \int_{V_1}^{V_2} P(V) dV \text{ by the Substitution Rule.}
\]

16. \( 160 \text{ lb/ft}^3 = 160 \cdot 144 \text{ lb/ft}^3, \quad 100 \text{ in}^3 = \frac{100}{1728} \text{ ft}^3, \quad \text{and} \quad 800 \text{ in}^3 = \frac{800}{1728} \text{ ft}^3.
\]

\[
k = PV^{1.4} = (160 \cdot 144) \left( \frac{100}{1728} \right)^{1.4} = 23,040 \left( \frac{25}{432} \right)^{1.4} \approx 426.5. \quad \text{Therefore,} \quad P \approx 426.5 V^{-1.4}
\]

\[
W = \int_{100/1728}^{800/1728} 426.5V^{-1.4} dV = 426.5 \left[ -\frac{1}{-0.4} V^{-0.4} \right]_{25/432}^{25/432} = (426.5)(2.5) \left( \left( \frac{25}{432} \right)^{0.4} - \left( \frac{25}{432} \right)^{0.4} \right)
\]

\[
\approx 1.88 \times 10^3 \text{ ft-lb}
\]
17. (a) \[ W = \int_a^b F(r) \, dr = \int_a^b \frac{G m_1 m_2}{r^2} \, dr = G m_1 m_2 \left( \frac{1}{a} - \frac{1}{b} \right) \]

(b) By part (a), \( W = G M m \left( \frac{1}{R} - \frac{1}{R + 1,000,000} \right) \) where \( M \) is mass of earth in kg, \( R \) is radius of earth in m, and \( m \) is mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

\[ W = \left( 6.67 \times 10^{-11} \right) \left( 5.98 \times 10^{24} \right) \left( 1000 \right) \times \left( \frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6} \right) \approx 8.50 \times 10^8 \, J. \]

18. (a) \[ W = \int_R^{\infty} \frac{G m M}{r^2} \, dr = \lim_{t \to \infty} \int_R^t \frac{G m M}{r^2} \, dr = \lim_{t \to \infty} G m M \left[ \frac{1}{r} \right]_R^t = G m m \lim_{t \to \infty} \left( \frac{1}{t} + \frac{1}{R} \right) = \frac{G m m}{R} \]

where \( M \) is mass of earth = 5.98 \times 10^{24} \, kg, \( m \) mass of satellite = 10^3 \, kg.

\( R \) is radius of earth = 6.37 \times 10^6 \, m, and \( G \) = gravitational constant = 6.67 \times 10^{-11} \, \text{N-m}^2/\text{kg}^2. \) Therefore, work = \( \frac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^8 \, J. \)

(b) From part (a), \( W = \frac{G m m}{R}. \) The initial kinetic energy supplies the needed work, so

\[ \frac{1}{2} m v^2 = \frac{G m m}{R} \quad \Rightarrow \quad v_0 = \sqrt{2 \frac{G m m}{R}}. \]

6.7 Probability

1. (a) \[ \int_{30,000}^{40,000} f(x) \, dx \] is the probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles.

(b) \[ \int_{25,000}^{26,000} f(x) \, dx \] is the probability that a randomly chosen tire will have a lifetime of at least 25,000 miles.

2. (a) The probability that you drive to school in less than 15 minutes is \[ \int_0^{15} f(t) \, dt. \]

(b) The probability that it takes you more than half an hour to get to school is \[ \int_{30}^{\infty} f(t) \, dt. \]

4. (a) As in the preceding exercise, (1) \( f(x) \geq 0 \) and

(2) \[ \int_{-\infty}^{\infty} f(x) \, dx = \int_0^{10} f(x) \, dx = \frac{1}{3}(10)(0.2) \] [area of a triangle] = 1. So \( f(x) \) is a probability density function.

(b) (i) \( P(X < 3) = \int_0^3 f(x) \, dx = \frac{1}{3}(3)(0.1) = \frac{1}{3} \approx 0.15 \)

(ii) We first compute \( P(X > 8) \) and then subtract that value from our answer in (i) (the total probability).

\[ P(X > 8) = \int_8^{10} f(x) \, dx = \frac{1}{2}(2)(0.1) = \frac{1}{10} = 0.10. \] So \( P(3 \leq X \leq 8) = 1 - 0.15 - 0.10 = 0.75. \)

(c) We find equations of the lines from \((0,0)\) to \((6,0.2)\) and from \((6,0.2)\) to \((10,0)\), and find that

\[ f(x) = \begin{cases} \frac{3}{5} x & \text{if } 0 \leq x \leq 6 \\ -\frac{1}{5} x + \frac{3}{5} & \text{if } 6 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases} \]

\[ \mu = \int_{-\infty}^{\infty} x f(x) \, dx = \int_0^6 x \left( \frac{3}{5} x \right) \, dx + \int_6^{10} x \left( -\frac{1}{5} x + \frac{3}{5} \right) \, dx = \left[ \frac{3}{20} x^4 \right]_0^6 + \left[ \frac{1}{20} x^5 + \frac{3}{4} x^2 \right]_6^{10} = \frac{216}{90} + \left( -\frac{1000}{60} + \frac{100}{4} \right) - \left( -\frac{216}{90} + \frac{36}{2} \right) = \frac{4}{3} = 1.33. \]

5. We need to find \( m \) so that \( \int_{-\infty}^{\infty} f(t) \, dt = \frac{1}{2} \) \( \Rightarrow \lim_{x \to \infty} f(x) \frac{1}{2} e^{-t/5} \, dt = \frac{1}{2} \) \( \Rightarrow \lim_{x \to \infty} \left[ \frac{1}{2} \left( 5 e^{-t/5} \right) \right]_m = \frac{1}{2} \) \( \Rightarrow \) \( -1 \left( 0 - e^{-m/5} \right) = \frac{1}{2} \) \( \Rightarrow e^{-m/5} = \frac{1}{2} \) \( \Rightarrow -m/5 = \ln 1/2 \) \( \Rightarrow m = -5 \ln 1/2 = 5 \ln 2 \approx 3.47 \) min.

7. We use an exponential density function with \( \mu = 2.5 \) min.

(a) \( P(X > 4) = \int_4^{\infty} f(t) \, dt = \lim_{x \to \infty} \int_0^x \frac{1}{2.5} e^{-t/2.5} \, dt = \lim_{x \to \infty} \left[ -e^{-t/2.5} \right]_x^0 = 0 + e^{-4/2.5} \approx 0.202 \)

(b) \( P(0 \leq X \leq 2) = \int_0^2 f(t) \, dt = \left[ -e^{-t/2.5} \right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551 \)

(c) We need to find a value \( a \) so that \( P(X \geq a) = 0.02, \) or, equivalently, \( P(0 \leq X \leq a) = 0.98 \) \( \Rightarrow \)

\[ f(t) \, dt = 0.98 \Leftrightarrow \left[ -e^{-t/2.5} \right]_0^a = 0.98 \Leftrightarrow e^{-a/2.5} + 1 = 0.98 \Leftrightarrow e^{-a/2.5} = 0.02 \Leftrightarrow -a/2.5 = \ln 0.02 \Leftrightarrow a = -2.5 \ln 1/2 = 2.5 \ln 2 \approx 9.78 \text{ min} \approx 10 \text{ min.} \] The ad should say that if you aren’t served within 10 minutes, you get a free hamburger.
8. (a) With \( \mu = 69 \) and \( \sigma = 2.8 \), we have \( P(65 \leq X \leq 73) = \int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} \exp\left(-\frac{(x - 69)^2}{2 \cdot 2.8^2}\right) \, dx \approx 0.847 \) (using a calculator or computer to estimate the integral).

(b) \( P(X > 6 \text{ feet}) = P(X > 72 \text{ inches}) = 1 - P(0 \leq X \leq 72) \approx 1 - 0.858 = 0.142 \), so 14.2\% of the adult male population is more than 6 feet tall.

9. \( P(X \geq 10) = \int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x - 9.4)^2}{2 \cdot 4.2^2}\right) \, dx \). To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, \( P(X \geq 10) \approx \int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x - 9.4)^2}{2 \cdot 4.2^2}\right) \, dx \approx 0.443 \) (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.

Note: We can’t evaluate \( 1 - P(0 \leq X \leq 10) \) for this problem since a significant amount of area lies to the left of \( X = 0 \).

11. \( P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \, dx \). Substituting \( t = \frac{x - \mu}{\sigma} \) and \( dt = \frac{1}{\sigma} \, dx \) gives us

\[ \int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt = \int_{-2}^{2} e^{-t^2/2} \, dt = 0.9545 \]

12. Let \( f(x) = \begin{cases} 0 & \text{if } x < 0 \\ cx e^{-cx} & \text{if } x \geq 0 \end{cases} \) where \( c = 1/\mu \). By using parts, tables, or a CAS, we find that

(1): \( \int x e^{bx} \, dx = \left( e^{bx}/b^2 \right) (bx - 1) \)

(2): \( \int x^2 e^{bx} \, dx = \left( e^{bx}/b^3 \right) (b^2 x^2 - 2bx + 2) \)

Now

\[ \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx = \int_{-\infty}^{0} (x - \mu)^2 f(x) \, dx + \int_{0}^{\infty} (x - \mu)^2 f(x) \, dx = 0 + \lim_{t \to \infty} \int_{t}^{0} (x^2 e^{-cx} - 2x\mu e^{-cx} + \mu^2 e^{-cx}) \, dx \]

Next we use (2) and (1) with \( b = -c \) to get

\[ \sigma^2 = c \lim_{t \to \infty} \left[ -\frac{e^{-cx}}{c^3} (c^2 x^2 + 2cx + 2) - 2\mu \frac{e^{-cx}}{c^2} (-cx - 1) + \mu^2 \frac{e^{-cx}}{c} \right]_t^0 \]

Using l’Hospital’s Rule several times, along with the fact that \( \mu = 1/c \), we get

\[ \sigma^2 = c \left[ 0 - \left( -\frac{2}{c^2} + \frac{2}{c} \cdot \frac{1}{c^2} + \frac{1}{c^2} \cdot \frac{1}{c} \right) \right] = c \left( \frac{1}{c^2} \right) = \frac{1}{c^2} \Rightarrow \sigma = \frac{1}{c} = \mu \]

13. (a) First \( p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0 \) for \( r \geq 0 \). Next,

\[ \int_{0}^{\infty} p(r) \, dr = \int_{0}^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0} \, dr = \frac{4}{a_0^3} \lim_{t \to \infty} \int_{0}^{t} r^2 e^{-2r/a_0} \, dr \]

As in Exercise 12, we use (2) from that solution (with \( b = -2/a_0 \)) and l’Hospital’s Rule to get

\[ \frac{4}{a_0^3} \left[ \frac{2}{a_0^2} (-2) \right] = 1. \]

This satisfies the second condition for a function to be a probability density function.

(b) Using l’Hospital’s Rule,

\[ \frac{4}{a_0^3} \lim_{r \to \infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r \to \infty} \frac{2r}{(2/a_0) e^{2r/a_0}} = \frac{2}{a_0^3} \lim_{r \to \infty} \frac{2}{(2/a_0) e^{2r/a_0}} = 0. \]

To find the maximum of \( p(r) \), we differentiate:

\[ p'(r) = \frac{4}{a_0^3} \left[ r e^{-2r/a_0} \left( -\frac{2}{a_0} \right) + e^{-2r/a_0} (2r^2) \right] = \frac{4}{a_0^3} e^{-2r/a_0} (2r) \left( -\frac{r}{a_0} + 1 \right) \]

\[ p'(r) = 0 \iff r = 0 \text{ or } 1 = \frac{r}{a_0} \iff r = a_0 \text{ [} a_0 \approx 5.59 \times 10^{-11} \text{ m]}. \]

\( p'(r) \) changes from positive to negative at \( r = a_0 \), so \( p(r) \) has its maximum value at \( r = a_0 \).

(e) \( \mu = \int_{-\infty}^{\infty} r p(r) \, dr = \frac{4}{a_0^3} \lim_{t \to \infty} \int_{0}^{t} r^3 e^{-2r/a_0} \, dr \). Integrating by parts three times or using a CAS, we find that

\[ \int x^3 e^{bx} \, dx = \frac{e^{bx}}{b^4} (b^2 x^3 - 3b^2 x^2 + 6bx - 6). \]

So with \( b = -\frac{2}{a_0} \), we use l’Hospital’s Rule, and get

\[ \mu = \frac{4}{a_0^3} \left[ \frac{a_0^4}{16} (-6) \right] = \frac{3}{8} a_0. \]