FINAL EXAM SOLUTIONS

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT

Good luck!

Name ________________________________
ID number __________________________

1. __________ (40 points)        “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

2. __________ (40 points)        Signature: __________________________

3. __________ (40 points)        Circle your TA’s name:

4. __________ (40 points)        Yonatan Gutman (2 and 6)

5. __________ (40 points)        Brett Parker (3 and 7)

Bonus __________ (20 points)       Yiannis Sakellaridis (4 and 8)

Circle your section meeting time:

Total __________ (200 points)       Ryan Vinroot (A02)

Michel Grueneberg (A03)

11:00am  1:15pm  7pm
1. (a) Write down, but do not evaluate, a Riemann sum that expresses the area between the $x$-axis and the graph of $f(x) = (1 + 2x)\sqrt{1 - x^2}$, between $x = 0$ and $x = 1$. Make sure that the only variables in your final answer are $i$ and $n$.

**Solution:** First we note that the function above is positive – so, we can use the standard Riemann sum

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

which in this case becomes

$$\lim_{n \to \infty} \sum_{i=1}^{n} (1 + 2x_i)\sqrt{1 - x_i^2}\Delta x$$

Since $a = 0$ and $b = 1$, we have that $\Delta x = (b - a)/n = 1/n$, and $x_i = a + i\Delta x = i/n$. The expression above then becomes

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)\sqrt{1 - \frac{i^2}{n^2}} \left(\frac{1}{n}\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{1}{n} + \frac{2i}{n^2}\right)\sqrt{1 - \frac{i^2}{n^2}}$$

(b) Express the quantity above instead as a single integral, and then use integral laws to rewrite that single integral as a sum of two simpler integrals.

**Solution:** Again since the function is positive, the integral is

$$\int_{0}^{1} f(x) \, dx = \int_{0}^{1} (1 + 2x)\sqrt{1 - x^2} \, dx$$

We can write this as the sum of two simpler integrals by distributing in the integrand, and then writing the ensuing integral of a sum as the sum of the two corresponding integrals.

$$\int_{0}^{1} (1 + 2x)\sqrt{1 - x^2} \, dx = \int_{0}^{1} \sqrt{1 - x^2} + 2x\sqrt{1 - x^2} \, dx$$

$$= \int_{0}^{1} \sqrt{1 - x^2} \, dx + \int_{0}^{1} 2x\sqrt{1 - x^2} \, dx$$
(c) Evaluate the integrals from part (b) by using the Fundamental Theorem for one of them, and interpreting the other as the area of a well-known geometric shape.

**Solution:** The first integral from part (b) we immediately recognize as representing the area of a quarter-circle of radius 1; so, it equals

\[
\int_0^1 \sqrt{1 - x^2} \, dx = \frac{1}{4} \pi (1)^2 = \frac{\pi}{4}
\]

For the second integral, we can find an antiderivative by making the substitution 
\( u = (1 - x^2) \), from which we can conclude 
\( du = -2x \, dx \). We get

\[
\int 2x \sqrt{1 - x^2} \, dx = \int u^{1/2} (-du) = \frac{-2}{3} u^{3/2} = \frac{-2}{3} (1 - x^2)^{3/2}
\]

So, the original integral becomes

\[
\int_0^1 2x \sqrt{1 - x^2} \, dx = \left[ \frac{-2}{3} (1 - x^2)^{3/2} \right]_0^1 = \frac{2}{3}
\]
2. Let \( R \) be the region that is to the right of the \( y \)-axis, and between the \( x \)-axis and the graph of the function

\[
f(x) = \left( 1 + \frac{1}{x + 1} \right) e^{-x}
\]

Define \( V \) to be the volume that is obtained by rotating the region \( R \) around the \( x \)-axis.

(a) Write down, but do not evaluate, an integral representing \( V \).

**Solution:** Since \( x \) is the independent variable, we slice up the area \( R \) vertically; after rotating, this then gives us thin cylindrical slices of radius \( f(x) \). So,

\[
V = \int dV = \int \pi (f(x))^2 \, dx = \pi \int_0^\infty \left( 1 + \frac{1}{x + 1} \right)^2 e^{-2x} \, dx
\]

(b) Use the Limit Comparison Theorem to determine whether this integral converges or diverges.

**Solution:** The “dominant part” of the integrand appears to be the exponential, since as \( x \) approaches infinity the rational factor approaches 1. So, we will attempt to apply the Limit Comparison Theorem by comparing with the convergent integral

\[
\int_0^\infty e^{-2x} \, dx
\]

Noting that the integrands of this and the original integral are both always positive, we see that we can apply the Theorem.

So, we need to consider the limit of the ratio of the two integrands. This is

\[
\lim_{x \to \infty} \left( \frac{\left( 1 + \frac{1}{x + 1} \right)^2 e^{-2x}}{e^{-2x}} \right) = \lim_{x \to \infty} \left( 1 + \frac{1}{x + 1} \right)^2 = \left( \lim_{x \to \infty} \left( 1 + \frac{1}{x + 1} \right) \right)^2 = 1
\]

This limit exists, is not zero, and is not infinity; so, by the Limit Comparison Theorem, the two integrals under consideration have the same convergence properties. And since we know that the comparison integral converges, we therefore conclude that

\[
\int_0^\infty \left( 1 + \frac{1}{x + 1} \right)^2 e^{-2x} \, dx
\]

converges.
(c) Suppose now that the volume $V$ is filled with a gas in such a way that the density $ho$ is not constant, but depends on $x$ according to

$$\rho(x) = \frac{e^{2x}}{(x + 1)^2}$$

Write down (but do not evaluate) an integral representing the total mass contained in the volume $V$, and determine whether that integral converges or diverges.

**Solution:**

$$m = \int dm = \int \rho(x) dV = \pi \int \rho(x)(f(x))^2 \, dx$$

$$= \pi \int_0^\infty \frac{e^{2x}}{(x + 1)^2} \left( \left(1 + \frac{1}{x+1}\right)e^{-x} \right)^2 \, dx = \pi \int_0^\infty \frac{(1 + \frac{1}{x+1})^2}{(x + 1)^2} \, dx$$

To determine whether this integral converges or not, we could actually evaluate an antiderivative and compute the limit directly. But we can also again try to use the Limit Comparison Theorem.

The “dominant part” of the integrand appears to be the denominator, since as we previously observed, the numerator approaches one. So, we use

$$\int_0^\infty \frac{1}{(x + 1)^2} \, dx$$

as our comparison integral.

Again, noting that the integrands of the original and comparison integrals are positive, we can try to use the Theorem. So we need to compute

$$\lim_{x \to \infty} \frac{(1 + \frac{1}{x+1})^2}{\frac{1}{(x+1)^2}} = \lim_{x \to \infty} \left(1 + \frac{1}{x+1} \right)^2 = 1$$

This integral exists, is not zero, and is not infinity; so, by the Limit Comparison Theorem, the integral we are interested in has the same convergence properties as the comparison integral, which we can easily check converges:

$$\int_0^\infty \frac{1}{(x + 1)^2} \, dx = \lim_{t \to \infty} \int_0^t \frac{1}{(x + 1)^2} \, dx$$

$$= \lim_{t \to \infty} \left[ -(x + 1)^{-1} \right]_0^t$$

$$= \lim_{t \to \infty} -(t + 1)^{-1} - (-1)$$

$$= 1$$

So, the integral above converges.
3. Consider the differential equation

\[ \frac{dy}{dx} = 2xy + e^{x^2} \]

(a) Show that the function \( y = xe^{x^2} \) is a solution to the above differential equation.

Solution: To check that the given function is a solution, we need to show that the differential equation is satisfied for all \( x \). In other words, we need to show that as functions, the left and right sided of the differential equations are the same.

The left side is

\[ \frac{dy}{dx} = \left( xe^{x^2} \right)' = (x)(2xe^{x^2}) + (1)(e^{x^2}) \]

The right side is

\[ 2xy + e^{x^2} = 2x \left( e^{x^2} \right) + e^{x^2} \]

These are equal as functions, and so \( y = xe^{x^2} \) is a solution to the above differential equation.
(b) Show that the solution which passes through the point $(0,1)$ cannot also pass through the point $(1,2)$.

**Solution:** We can check easily that for the function $f(x) = xe^{x^2}$, we have $f(0) = 0$. From this, we can conclude that the point $(0,1)$ is above the graph of this function $f(x)$.

Also, since the equation above is in the form of a slope field, we can invoke the theorem that says that solutions to differential equations with slope fields cannot cross. From this, we conclude that the graph of the solution which passes through the point $(0,1)$ must always be above the graph of function $f(x)$.

However, since $f(1) = e$, and $2 < e$, we can also easily see that the point $(1,2)$ is below the graph of the function $f(x)$. Therefore, it is not possible for it to be on the solution discussed above.
(c) Find the general solution to the differential equation

\[ \frac{dy}{dx} = x^2 y \]

**Solution:** This equation is separable; so we can separate it and then antidifferentiate each side.

\[ \frac{dy}{dx} = x^2 y \]

\[ \frac{dy}{y} = x^2 \, dx \]

(We must later check the case \( y = 0 \))

\[ \int \frac{dy}{y} = \int x^2 \, dx \]

\[ \ln |y| = \frac{1}{3} x^3 + c \]

\[ |y| = e^c e^{\frac{1}{3} x^3} \]

\[ y = (\pm e^c) e^{\frac{1}{3} x^3} \]

Since \( e^c \) can be any positive number, we can write this as

\[ y = Ae^{\frac{1}{3} x^3} \quad (A \neq 0) \]

Of course as noted above, we also need to see what happens if \( y = 0 \); simply plugging in to the differential equation reveals that this is also a solution. And of course, the function \( y = 0 \) is exactly what we would get if we used \( A = 0 \) in the above form.

So, the complete general solution is

\[ y = Ae^{\frac{1}{3} x^3} \]

where \( A \) can be any real number.
(d) Now consider the “rabbits and wolves” system of differential equations

\[
\frac{dR}{dt} = kR - aRW \\
\frac{dW}{dt} = -rW + bRW
\]

Letting \( R \) be the horizontal axis and \( W \) be the vertical axis, sketch a diagram of the first quadrant indicating at which points the slope of a solution must be vertical, and at which points the slope of a solution must be horizontal. For what values of \( R \) and \( W \) would the populations both be constant? Explain your reasoning.

**Solution:** The slope must be horizontal at any point where \( dW/dt = 0 \). This gives us

\[ W(-r + bR) = 0 \iff W = 0 \text{ or } R = \frac{r}{b} \]

Similarly, the slope must be vertical wherever \( dR/dt = 0 \). This gives us

\[ R(k - aW) = 0 \iff R = 0 \text{ or } W = \frac{k}{a} \]

Those are all pictured on the figure on the next page.

Both populations would be constant at any point where both of the derivatives are zero; given our previous arguments, this happens at any point where the two sets represented above intersect. This happens at the point \((0,0)\), and the point \((r/b, k/a)\)

Those points are also represented on the figure on the next page.
$W = \frac{k}{a}$

$R = \frac{r}{b}$

$(r/b, k/a)$

$(0,0)$

$R = \frac{r}{b}$
4. (a) Use the ratio test to show that the Taylor series for $e^x$ converges for all values of $x$.

**Solution:** The Taylor series for $e^x$ is given by

\[ t(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \]

To apply the Ratio Test, we must compute the limit of the absolute value of the ratio of successive terms; this gives us

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0
\]

This limit is less than one, and so the Ratio Test tells us that this series converges. (Note however that this argument alone does NOT tell us that the Taylor series for $e^x$ actually converges to $e^x$...)

(b) Derive the Taylor series for the function $f(x) = \sin(2x)$ directly from the definition of the Taylor series (do NOT just make a substitution in the Taylor series for $\sin x$). You do not need to show that it converges.

**Solution:** First we compute a few derivatives of $f$:

\[
\begin{align*}
f^{[0]}(x) &= \sin(2x) \implies f^{[0]}(0) = 0 \\
f^{[1]}(x) &= 2 \cos(2x) \implies f^{[1]}(0) = 2 \\
f^{[2]}(x) &= -2^2 \sin(2x) \implies f^{[2]}(0) = 0 \\
f^{[3]}(x) &= -2^3 \cos(2x) \implies f^{[3]}(0) = -2^3 \\
f^{[4]}(x) &= 2^4 \sin(2x) \implies f^{[4]}(0) = 0 \\
f^{[5]}(x) &= 2^5 \cos(2x) \implies f^{[5]}(0) = 2^5 \\
&\vdots
\end{align*}
\]

From this we see that $f^{[n]}(0)$ is zero for all even values of $n$; so, those terms in the Taylor series are all zero. We are left only with the odd numbered terms. Plugging in to the form for the Taylor series, we get

\[
T(x) = f(0) + f'(0)(x) + \frac{f^{[2]}(0)x^2}{2!} + \frac{f^{[3]}(0)x^3}{3!} + \frac{f^{[4]}(0)x^4}{4!} + \frac{f^{[5]}(0)x^5}{5!} + \ldots
\]

\[
= 0 + 2x + 0 + \frac{-2^3x^3}{3!} + 0 + \frac{2^5x^5}{5!} + \ldots
\]

\[
= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \ldots
\]
(c) Suppose that a function $g(x)$ has the property that all of its derivatives exist for all values of $x$, and furthermore that each derivative is bounded:

$$g^{[n]}(x) \leq n^2 \quad \text{for all } x, n$$

Use Taylor’s Inequality to prove that the Taylor series for $g(x)$ converges to $g(x)$ for all values of $x$.

**Solution:** We show that the Taylor series converges to $g(x)$ by showing that the errors $R_n(x)$ must approach zero, for all values of $x$. To do this, we make use of the Taylor Inequality.

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{(n+1)}$$

In this case we are already given bounds on $g^{[n]}$; so, we can just use this as our $M$. So, we have

$$|R_n(x)| \leq \frac{n^2}{(n+1)!} |x - a|^{(n+1)}$$

To show that $R_n(x)$ approaches zero, it is enough to show that the expression on the right above approaches zero. To see this, we apply the Ratio Test for Sequences, for which we must compute the limit below:

$$\lim_{n \to \infty} \left| \frac{(n + 1)^2 |x - a|^{n+2}/(n + 2)!}{n^2 |x - a|^{n+1}/(n + 1)!} \right|$$

Of course most of this cancels... we are left with

$$\lim_{n \to \infty} \left| \frac{(n + 1)^2 |x - a|}{n^2(n + 2)} \right| = |x - a| \lim_{n \to \infty} \frac{(n + 1)^2}{n^2(n + 2)}$$

$$= |x - a| \lim_{n \to \infty} \frac{1 + 2/n + 1/n^2}{(n + 2)}$$

This limit equals zero, because the numerator approaches a constant and the denominator goes to infinity. Since it is zero, it is definitely less than one.

So, the Ratio Test for Sequences tells us that

$$\lim_{n \to \infty} \left| \frac{(n + 1)^2 |x - a|^{n+2}/(n + 2)!}{n^2 |x - a|^{n+1}/(n + 1)!} \right| = 0$$

Therefore the errors $R_n(x)$ must approach zero, for all values of $x$, and so the Taylor series converges to the function $g(x)$. 

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5. It can be shown the the convergence or divergence of the improper integral

$$\int_2^\infty \frac{\sin(\pi x)}{\ln x} \, dx$$

is completely determined by the convergence or divergence of the series

$$\sum_{n=2}^\infty a_n \quad \text{where} \quad a_n = \int_n^{n+1} \frac{\sin(\pi x)}{\ln x} \, dx$$

(a) Use a substitution and elementary properties of trig functions to show that

$$a_n = (-1)^n \int_0^1 \frac{\sin(\pi x)}{\ln(n+x)} \, dx$$

**Solution:** The bounds for the given and desired integrals suggest that perhaps a good substitution to try would be \(u = x - n\); making this substitution, we get

$$a_n = \int_n^{n+1} \frac{\sin(\pi x)}{\ln x} \, dx = \int_0^1 \frac{\sin(\pi(n+u))}{\ln(n+u)} \, du = \int_0^1 \frac{\sin((\pi u) + n\pi)}{\ln(n+u)} \, du$$

To simplify the numerator of the integrand, we can make use of the trig identity \(\sin(\theta + \pi) = -\sin(\theta)\). Applying this to the numerator \(n\) times, we get

$$a_n = \int_0^1 \frac{\sin((\pi u) + n\pi)}{\ln(n+u)} \, du = \int_0^1 \frac{(-1)^n \sin(\pi u)}{\ln(n+u)} \, du$$

$$= (-1)^n \int_0^1 \frac{\sin(\pi u)}{\ln(n+u)} \, du$$

(b) Show that \(|a_n| \leq \frac{1}{\ln n}\), and conclude from this that \(\lim_{n \to \infty} a_n = 0\).

**Solution:** Since the integrand in the result for part (a) is always positive, we get \(|a_n|\) by simply removing the powers of \((-1)\). Then, notice that \(\sin(\pi x) \leq 1\), and \(\ln(n+x) \geq \ln(n)\). This allows us to conclude that

$$|a_n| = \int_0^1 \frac{\sin(\pi x)}{\ln(n+x)} \, dx \leq \int_0^1 \frac{1}{\ln n} \, dx = \frac{1}{\ln n}$$

And since \(\lim_{n \to \infty} (1/\ln n) = 0\), we must have that \(\lim_{n \to \infty} |a_n| = 0\), which also gives us \(\lim_{n \to \infty} a_n = 0\).
(c) Show that $|a_{n+1}| < |a_n|$ for all $n \geq 2$.

**Solution:** Again, we must note that

$$|a_n| = \int_0^1 \frac{\sin(\pi x)}{\ln(n + x)} \, dx$$

Since all of the terms in the integrand are always positive, we can reason as follows:

$$\ln(n + 1 + x) > \ln(n + x)$$

$$\frac{1}{\ln(n + 1 + x)} < \frac{1}{\ln(n + x)}$$

$$\frac{\sin(\pi x)}{\ln(n + 1 + x)} < \frac{\sin(\pi x)}{\ln(n + x)}$$

$$\int_0^1 \frac{\sin(\pi x)}{\ln(n + 1 + x)} \, dx < \int_0^1 \frac{\sin(\pi x)}{\ln(n + x)} \, dx$$

Therefore $|a_{n+1}| < |a_n|$.

(d) Use the previous two conclusions about the series $\sum a_n$, along with another observation, to determine whether the series (and thus also the original integral) converges or diverges. Make sure to explain all of your reasoning.

**Solution:** The conclusions of parts (b) and (c) are two of the three requirements for the Alternating Series Test. The third is to verify that the terms actually do alternate in sign.

This follows from part (a), though, because the integrand of that integral is always positive.

So, $\sum a_n$ is a sum of alternating terms, decreasing in absolute value, and converging to zero; the Alternating Series Test then tells us that the series must converge. And due to the original assertion about the relationship between the given series and the given integral, we conclude that the original integral also converges.

**Side note:** The original integral in question is interesting because NONE of the convergence tests discussed in Chapter 5 can be used to determine whether it converges or diverges.
**Bonus Question:** Suppose that the series
\[ \sum_{n=0}^{\infty} a_n \]
converges. Prove that the power series
\[ \sum_{n=0}^{\infty} a_n x^n \]
must converge absolutely on the interval \((-1, 1)\).

**Solution:** If the series
\[ \sum_{n=0}^{\infty} a_n \]
converges, then we know that as a sequence, \( \{a_n\}_n \to 0 \).

Recalling the definition of the limit of a sequence (and using \( \epsilon = 1 \)), we can conclude that there must exist some \( N \) for which \( n > N \implies |a_n| < 1 \).

Of course, to show that the given power series converges absolutely, it is enough to show that
\[ \sum_{n=N+1}^{\infty} |a_n x^n| \]
converges, since this only ignores a finite number of terms in the series we get from the definition of absolute convergence. Since for this series we have \( n > N \), we can conclude that
\[ |a_n| < 1 \quad \text{and therefore that} \quad |a_n||x^n| < |x^n| \]

So, by the Comparison Theorem, we see that it is enough to check that
\[ \sum_{n=N+1}^{\infty} |x^n| = \sum_{n=N+1}^{\infty} |x|^n \]
converges.

Of course, this is just a geometric series with ratio \(|x|\); and for values of \( x \) in the interval \((-1, 1)\), that ratio is less than one. So, the geometric series converges, and by the Comparison Theorem, the previous series converges, which shows that the original power series converges absolutely.