5.9 The goal is to find scalars $a$ and $b$ such that

$$a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This leads to the system of equations

$$2a + 3b = 1$$
$$1a + 2b = 2$$

The second equation implies $a = 2 - 2b$, so inserting this into the first equation gives $4 - 4b + 3b = 1$, or $b = 3$. Plugging back into either equation gives $a = -4$, so

$$-4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

5.10 We want to know if there exist scalars $c_1, c_2, c_3$ such that

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -11 \end{bmatrix}$$

This leads to the system of equations

$$2c_1 + 5c_2 + c_3 = 8$$
$$c_1 + 8c_2 + 6c_3 = -5$$
$$c_1 - 2c_3 = -11$$

which has augmented matrix

$$A = \begin{bmatrix} 2 & 5 & 1 & 8 \\ 1 & 8 & 6 & -5 \\ 1 & 0 & -2 & -11 \end{bmatrix}$$

Since

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the last equation of the reduced system is $0 = 1$, so there are no solutions. Thus $\textbf{v}$ is not in the span of the given set of vectors.

5.11 We need to find a solution to the system of equations that corresponds to the following matrix of coefficients, which we row reduce:
\[
\begin{bmatrix}
2 & 5 & 11 & 8 \\
1 & 8 & 2 & -5 \\
1 & 0 & -12 & -11
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -12 & -11 \\
2 & 5 & 11 & 8 \\
1 & 8 & 2 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -12 & -11 \\
0 & 5 & 35 & 30 \\
0 & 8 & 14 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -12 & -11 \\
0 & 1 & 7 & 6 \\
0 & 4 & 7 & 3
\end{bmatrix}
\]

So we conclude that

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
5 \\
8 \\
0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
11 \\
2 \\
-12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
8 \\
-5 \\
-11
\end{bmatrix}
\]

5.12 We want to solve

\[
c_1 \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
+ c_2 \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
+ c_3 \begin{bmatrix}
4 \\
3 \\
2
\end{bmatrix}
= \begin{bmatrix}
1 \\
5 \\
9 \\
13
\end{bmatrix}
\]

The augmented matrix corresponding to this system is

\[
A =
\begin{bmatrix}
1 & 1 & 4 & | & 1 \\
1 & 2 & 4 & | & 5 \\
1 & 3 & 2 & | & 9 \\
1 & 4 & 1 & | & 13
\end{bmatrix}
\]

Since

\[
\text{rref}(A) =
\begin{bmatrix}
1 & 0 & 5 & | & -3 \\
0 & 1 & -1 & | & 4 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]

the reduced system is

\[
c_1 + 5c_3 = -3 \\
\]

\[
c_2 - c_3 = 4
\]

Choosing \(c_3 = 0\) we get \(c_1 = -3\) and \(c_2 = 4\). Thus

\[
\begin{bmatrix}
1 \\
5 \\
9 \\
13
\end{bmatrix}
= -3 \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
+ 4 \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}
+ 0 \begin{bmatrix}
4 \\
3 \\
2 \\
1
\end{bmatrix}
\]

There are infinitely many solutions — every choice of \(c_3\) yields a different linear combination which equals \(v\).
6.1 The augmented matrix is
\[
\begin{bmatrix}
1 & 0 & 1 & 0 & | & 5 \\
1 & 2 & 3 & 4 & | & 13 \\
1 & 2 & 1 & 2 & | & 5 \\
\end{bmatrix}
\]

Its reduced row echelon form is
\[
\begin{bmatrix}
1 & 0 & 0 & -1 & | & 1 \\
0 & 1 & 0 & 1 & | & 0 \\
0 & 0 & 1 & 1 & | & 4 \\
\end{bmatrix}
\]

There is no inconsistency, and \( z \) is a free variable, so there exist infinitely many solutions.

6.2 The augmented matrix is
\[
\begin{bmatrix}
1 & 2 & 3 & | & 1 \\
2 & 1 & -2 & | & 1 \\
\end{bmatrix}
\]

Its reduced row echelon form is
\[
\begin{bmatrix}
1 & 2 & 3 & | & 1 \\
0 & 1 & 8/3 & | & 1/3 \\
\end{bmatrix}
\]

There is no inconsistency, and \( z \) is a free variable, so there exist infinitely many solutions.

6.3 The augmented matrix is
\[
\begin{bmatrix}
1 & -2 & 1 & | & 0 \\
2 & 2 & -1 & | & 8 \\
3 & 1 & 2 & | & 0 \\
1 & -2 & 3 & | & -7 \\
\end{bmatrix}
\]

Its reduced row echelon form is
\[
\begin{bmatrix}
1 & 0 & 0 & | & 0 \\
0 & 1 & 0 & | & 0 \\
0 & 0 & 1 & | & 0 \\
0 & 0 & 0 & | & 1 \\
\end{bmatrix}
\]

Thus the final equation of the reduced system is \( 0 = 1 \), which means there are no solutions.

6.7 (a) After switching rows 1 and 2, the reduced row echelon form is
\[
\begin{bmatrix}
-1 & 4 & 0 & | & 0 \\
0 & 11 & 1 & | & 0 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

Here \( z \) is a free variable and the solutions are \((-4z/11, -z/11, z)\).
(b) Again we switch rows 1 and 2, and the reduced row echelon form is

$$\begin{bmatrix}
-1 & 4 & 0 & | & 1 \\
0 & 11 & 1 & | & -1 \\
0 & 0 & 0 & | & -2 \\
\end{bmatrix}$$

The last row says $0 = -2$, so the system is inconsistent. No solutions.

6.13 From the corresponding equations $p(1) = 1$, $p(2) = 2$, and $p(-1) = 5$ we get the equations

\begin{align*}
a + b + c &= 1 \\
4a + 2b + c &= 2 \\
a - b + c &= 5
\end{align*}

The augmented matrix for this system is

$$\begin{bmatrix}
1 & 1 & 1 & | & 1 \\
4 & 2 & 1 & | & 2 \\
1 & -1 & 1 & | & 5
\end{bmatrix}$$

with reduced row echelon form

$$\begin{bmatrix}
1 & 1 & 1 & | & 1 \\
0 & -2 & -3 & | & -2 \\
0 & 0 & 3 & | & 6
\end{bmatrix}$$

So the solutions are $c = 2$, $b = -2$, and $a = 1$, and the polynomial is $p(x) = x^2 - 2x + 2$.

6.14 Substituting the given information, we obtain the following system

\begin{align*}
-8 + 4a - 2b + c &= 2 \\
-1 + a - b + c &= 3 \\
1 + a + b + c &= 0 \\
8 + 4a + 2b + c &= 8
\end{align*}

Moving the constant terms to the right hand side, we obtain a system whose augmented matrix is

$$\begin{bmatrix}
4 & -2 & 1 & | & 10 \\
1 & -1 & 1 & | & 4 \\
1 & 1 & 1 & | & -1 \\
4 & 2 & 1 & | & 0
\end{bmatrix}$$

The reduced row echelon form is

$$\begin{bmatrix}
1 & 0 & 0 & | & 7/6 \\
0 & 1 & 0 & | & -5/2 \\
0 & 0 & 1 & | & 1/3 \\
0 & 0 & 0 & | & 0
\end{bmatrix}$$
Thus the unique solution is \( a = \frac{7}{6}, \ b = -\frac{5}{2}, \ c = \frac{1}{3} \), which means

\[
p(x) = x^3 + \frac{7}{6}x^2 - \frac{5}{2}x + \frac{1}{3}
\]
satisfies all of the given conditions. (Check!)

6.15 We have

\[
\begin{align*}
f(t) &= A \cos t + B \sin t + Ce^t \\
f'(t) &= -A \sin t + B \cos t + Ce^t \\
f''(t) &= -A \cos t - B \sin t + Ce^t
\end{align*}
\]

so the three conditions \( f(0) = 2, \ f'(0) = 0 \) and \( f''(0) = 6 \) lead to the system

\[
\begin{align*}
A + C &= 2 \\
B + C &= 0 \\
-A + C &= 6
\end{align*}
\]

The reduced row echelon form of the system is

\[
\begin{align*}
A &= -2 \\
B &= -4 \\
C &= 4
\end{align*}
\]

so \( A = -2, \ B = -4 \) and \( C = 4 \).

8.10 The observation to make here is that \( x = c \) is a solution of \( Ax = Ac \). By Proposition 11.1, every other solution takes the form \( c + x_n \) where \( x_n \in N(A) \). The null space was found in Exercise 64.

(a) \( x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \)

(b) \( x = \begin{bmatrix} -3 \\ 2 \\ 4 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \)

(c) \( x = \begin{bmatrix} \pi/6 \\ \sqrt{e}/2 \\ 195 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \)

8.12 Since \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \)
the system \( Ax = 0 \) reduces to

\[
\begin{align*}
    x_1 + 2x_3 &= 0 \\
    x_2 + x_3 - x_4 &= 0
\end{align*}
\]

The solutions are

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = x_3 \begin{bmatrix}
    -2 \\
    1 \\
    1 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    0 \\
    1 \\
    0 \\
    1
\end{bmatrix}
\]

so

\[
N(A) = \text{span} \left( \begin{bmatrix}
    -2 \\
    -1 \\
    1 \\
    0
\end{bmatrix}, \begin{bmatrix}
    0 \\
    1 \\
    0 \\
    1
\end{bmatrix} \right)
\]

We can find a particular solution to the original equation by setting \( x_3 = x_4 = 0 \); solving for the remaining variables, we get \( x_1 = 9 \) and \( x_2 = 2 \). So, the complete set of solutions in parametric form is

\[
\begin{bmatrix}
    9 \\
    2 \\
    0 \\
    0
\end{bmatrix} + x_3 \begin{bmatrix}
    -2 \\
    -1 \\
    1 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    0 \\
    1 \\
    0 \\
    1
\end{bmatrix}
\]

8.13 The augmented matrix for this system is

\[
\begin{bmatrix}
    1 & 3 & -1 & 9 & 7 \\
    1 & 1 & 3 & 1 & 9 \\
    2 & 7 & -4 & 22 & 13
\end{bmatrix}
\]

Its reduced row echelon form is

\[
\begin{bmatrix}
    1 & 0 & 5 & -3 & 10 \\
    0 & 1 & -2 & 4 & -1 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

so the reduced form of the system is

\[
\begin{align*}
    x_1 + 5x_3 - 3x_4 &= 10 \\
    x_2 - 2x_3 + 4x_4 &= -1 \\
    0 &= 0
\end{align*}
\]

The solutions are therefore

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    10 - 5x_3 + 3x_4 \\
    -1 + 2x_3 - 4x_4 \\
    x_3 \\
    x_4
\end{bmatrix} = x_3 \begin{bmatrix}
    10 \\
    -1 \\
    0 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    -5 \\
    2 \\
    1 \\
    0
\end{bmatrix} + \begin{bmatrix}
    3 \\
    4 \\
    0 \\
    1
\end{bmatrix}
\]
9.2 Performing elimination on the augmented matrix

\[
\begin{bmatrix}
1 & 3 & -1 & 9 & b_1 \\
1 & 1 & 3 & 1 & b_2 \\
2 & 7 & -4 & 22 & b_3 \\
\end{bmatrix}
\]

we arrive at

\[
\begin{bmatrix}
1 & 3 & -1 & 9 & b_1 \\
0 & 1 & -2 & 4 & \frac{b_1}{2} - \frac{b_2}{2} \\
0 & 0 & 0 & 0 & \frac{5b_1}{2} - \frac{b_2}{2} - b_3 \\
\end{bmatrix}
\]

Although this is not the reduced row echelon form, we need not continue further, since we can already see where the inconsistencies could possibly arise. From the third row, we see that the components of \( \mathbf{b} \) must satisfy

\[
\frac{5b_1}{2} - \frac{b_2}{2} - b_3 = 0
\]

to avoid inconsistency.

9.3 Performing elimination on the augmented matrix

\[
\begin{bmatrix}
1 & 1 & b_1 \\
-2 & 1 & b_2 \\
0 & 3 & b_3 \\
2 & -1 & b_4 \\
\end{bmatrix}
\]

we arrive at

\[
\begin{bmatrix}
1 & 1 & b_1 \\
0 & 1 & \frac{1}{3}b_3 \\
0 & 0 & b_2 + 2b_1 - b_3 \\
0 & 0 & b_4 - 3b_1 + b_3 \\
\end{bmatrix}
\]

Although this is not the reduced row echelon form, we need not continue further, since we can already see where the inconsistencies could possibly arise. From the third and fourth rows, we see that the components of \( \mathbf{b} \) must satisfy

\[
2b_1 + b_2 - b_3 = 0 \\
-2b_1 + b_3 + b_4 = 0
\]

to avoid inconsistency.

9.4 (b) Since \( 2(0) + 3 - 3 = 0 \) and \( -2(0) + 3 + (-3) = 0 \), \[
\begin{bmatrix}
0 \\
3 \\
3 \\
-3 \\
\end{bmatrix}
\]
is in the column space of \( A \).
(c) Since $2(1) + 2 - 4 = 0$ and $-2(1) + 4 + (-2) = 0$, \[
\begin{bmatrix}
1 \\
2 \\
4 \\
-2
\end{bmatrix}
\] is in the column space of $A$.

(e) Since $-2(2) + 3 + (-1) = -2 \neq 0$, \[
\begin{bmatrix}
2 \\
1 \\
3 \\
-1
\end{bmatrix}
\] is not in the column space of $A$. 