Linearity of the Change of Variable Function

In class, we pointed out that a vector $\vec{x}$ can be described with a list of numbers (understood to be the coefficients of some linear combination of the vectors in a basis) ONLY if we make clear which basis we are using. For example, we can write a vector $\vec{x}$ as:

\[
[\vec{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \iff \vec{x} = x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n \text{ where } S = \{ \vec{e}_1, \ldots, \vec{e}_n \}
\]

or

\[
[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \iff \vec{x} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n \text{ where } B = \{ \vec{v}_1, \ldots, \vec{v}_n \}
\]

Since we can write a vector as a list of numbers in different ways, we might want to have an easy way to switch between these different representations.

In other words, we want a function $f$ that takes the coefficients of some vector $\vec{x}$ in terms of the basis $B$, and returns the coefficients of the SAME vector in terms of the basis $S$. For example, using the example above, we would have $f(c_1, \ldots, c_n) = (x_1, \ldots, x_n)$.

It is going to turn out that this function is in fact a matrix. We will eventually write down this matrix, but before we can do that we have to show that in fact the function is linear.

This claim seems to have been the source of some confusion in class, so I would like make the following more explicit demonstration...

**Claim:** The function $f$ described above is linear – namely,

\[
f \left( (c_1, \ldots, c_n) + (c'_1, \ldots, c'_n) \right) = f(c_1, \ldots, c_n) + f(c'_1, \ldots, c'_n)
\]

and

\[
f \left( k \cdot (c_1, \ldots, c_n) \right) = k \cdot f(c_1, \ldots, c_n)
\]

**Proof:** Let’s start with the first equation; the left hand side is

\[
f \left( (c_1, \ldots, c_n) + (c'_1, \ldots, c'_n) \right)
\]

On the inside, we have the sum of two sets of coefficients. If individually these are the coefficients (with respect to the $B$ basis) of vectors $\vec{x}$ and $\vec{x}'$, then their sum is the list of coefficients (with respect to the $B$ basis) of the vector $\vec{x} + \vec{x}'$. 
So, we have

\[ = f \left( \text{coefficients of } \vec{x} + \vec{x}' \text{ with respect to the } \mathcal{B} \text{ basis} \right) \]

Of course, the only thing that the function \( f \) does is to change the coefficients to represent the SAME vector, in terms of the other basis, \( \mathcal{S} \). So we have

\[
= \left( \text{coefficients of } \vec{x} + \vec{x}' \text{ with respect to the } \mathcal{S} \text{ basis} \right) \\
= \left( \text{coefficients of } \vec{x} \text{ with respect to the } \mathcal{S} \text{ basis} \right) \\
+ \left( \text{coefficients of } \vec{x}' \text{ with respect to the } \mathcal{S} \text{ basis} \right)
\]

Now that we have separated this into two distinct terms, we can rewrite back in terms of the function \( f \).

\[
= f \left( \text{coefficients of } \vec{x} \text{ with respect to the } \mathcal{B} \text{ basis} \right) \\
+ f \left( \text{coefficients of } \vec{x}' \text{ with respect to the } \mathcal{B} \text{ basis} \right)
\]

and this is what we wanted to compute in the first place

\[ = f \left( c_1, \ldots, c_n \right) + f \left( c'_1, \ldots, c'_n \right) \]

The proof for the second equation is similar – try this as an exercise for yourself once you feel comfortable with the derivation above.

In some sense, the thing that makes this all work is the fact that really, nothing is happening here. Basically, the computation just says that when you add two vectors, the coefficients add – and more specifically, that this is true with respect to either basis. That’s about all that is happening in this computation.

I don’t want to make too big of a deal about this point, since I think it should be sort of believable that this function should be linear. So, I won’t spend any more time on this question in class. But if you are still unhappy with the claim and would like to discuss it further, feel free to ask me, or the TA’s, or a friend.