EXAM II SOLUTIONS


You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT

Good luck!

Name ________________________________

ID number ____________________________

1. ___________ (/20 points)  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

2. ___________ (/20 points)  Signature: ____________________________

3. ___________ (/20 points)  Circle your TA’s name:

4. ___________ (/20 points)

   Mohsen Bayati (2 and 6)

   Byoung-du Kim (3 and 7)

5. ___________ (/20 points)

   Olivier Daviaud (4 and 8)

   Heaseung Kwon (A02)

   Alex Meadows (A03)

Bonus ___________ (/10 points)

Circle your section meeting time:

Total ___________ (/100 points)  11:00am  1:15pm  7pm
1. Compute the volume of the solid obtained by rotating, around the line $y = -1$, the region bounded by $x = y^2 - y^3$ and the $y$-axis. (Hint: What is the independent variable?)

Solution: The independent variable is clearly $y$ in the equation given, so we slice up the indicated area horizontally, as shown in the diagram above. When we rotate these slices of area around $y = -1$, they form cylindrical shells. So we get

$$V = \int dV = \int_0^1 (h)(2\pi \text{radius}) \, dy$$

The height $h$ is $x$; the radius is just $y - (-1)$. So, our integral becomes

$$= \int_0^1 (h)(2\pi \text{radius}) \, dy = \int_0^1 (x)(2\pi(y + 1)) \, dy = \int_0^1 (y^2 - y^3)(2\pi(y + 1)) \, dy$$

$$= 2\pi \int_0^1 -y^4 + y^2 \, dy = 2\pi \left( -\frac{y^5}{5} + \frac{y^3}{3} \right)_0 = \frac{4\pi}{15}$$
2. The acceleration due to the gravitational attraction of the moon (as a function of the distance $x$ to the center of the moon) is given by

$$a = \frac{-k}{x^2}$$

where $k$ is a constant. Using this information, determine the amount of work it would take to pull a body of mass $m$, initially on the surface of the moon, completely away from the moon's gravitational field (for the purposes of the integral you will set up, this means that we want to pull it an infinite distance from the center of the moon).

Your answer will depend on the gravitational constant ($k$), the radius of the moon ($R$), and the mass of the body ($m$). Recall that $F = ma$, and $W = Fd$.

![Diagram of moon and distance $x$]

**Solution:** As is indicated in the diagram above, we break up the problem by breaking up the distance. Our integral is

$$W = \int dW = \int_{R}^{\infty} F \, dx = \int_{R}^{\infty} (ma) \, dx$$

The gravitational acceleration $a$ is given to us in terms of $x$; of course, the force we apply is opposite to the force of gravity, and thus its negative. So, the integral becomes

$$= \int_{R}^{\infty} (ma) \, dx = \int_{R}^{\infty} \frac{mk}{x^2} \, dx$$

$$= \lim_{t \to \infty} \int_{R}^{t} \frac{mk}{x^2} \, dx = \lim_{t \to \infty} \left( \frac{-mk}{x} \right)_{R}^{t} = \lim_{t \to \infty} \left( \frac{mk}{R} - \frac{mk}{t} \right) = \frac{mk}{R}$$
3. (For this problem, use the approximations $\frac{1}{1-e^{-\pi}} \approx 1.05$, and $(1 - e^{-1}) \approx .63$)

   (a) In order to pay off a gambling debt of $100,000$, you are forced to borrow money from a loan shark, who insists on charging you the outrageous interest rate of 30% per year (compounded continuously). You agree to pay off the loan over ten years.

   What is your (continuous) payment rate, in dollars per year?

   **Solution:** We need for the present value of the promised payment stream to equal $100,000$. Let the pay rate be $p$ dollars per year. We then have

   $$\int_0^{10} (\text{pay rate}) \, dt = \int_0^{10} p \, dt = \int_0^{10} p \left( e^{r(0-t)} \right) \$0 \, dt$$

   $$100,000 = p \int_0^{10} (e^{-3t}) \, dt = \frac{-p}{3} (e^{-3t}) \bigg|_0^{10} = \frac{p}{3} \left( 1 - e^{-3} \right)$$

   $$p = 30,000 \left( \frac{1}{1 - e^{-3}} \right) \approx 30,000(1.05) \approx 31,500$$

   So, the pay rate is $31,500 per year.

   (b) Assuming that the true fair interest rate would be 10% per year, compute the present value of the payment stream you have agreed to pay the loan shark.

   **Solution:** The value of a payment stream of $31,500 per year for ten years is the following integral:

   $$\int_0^{10} 31,500 \$t \, dt$$

   Since we are interested in present value, we want to evaluate this in terms of current dollars, or $0$’s. So the integral becomes

   $$= \int_0^{10} 31,500 \left( e^{r(0-t)} \right) \$0 \, dt = 31,500 \$0 \int_0^{10} \left( e^{(-.01t)} \right) \, dt$$

   $$= 31,500 \$0 \left( -10e^{(-.00)} \right) \bigg|_0^{10} = 315,000 \$0 \left( (1 - e^{-1}) \right)$$

   $$\approx 315,000 \$0 (0.63) \approx 198,450 \$0$$
4. (a) At $t = 0$, a pot containing 50L of pure water begins leaking at the bottom. The rate of leaking is proportional to the volume of water remaining in the pot; and at time $t = 0$, the rate is 10L per minute.

Find the expression for the volume of water remaining in the pot, as a function of time.

**Solution:** Let $V(t)$ be the volume of water remaining in the pot at a time $t$. Then, the statement about the rate of leaking gives us the differential equation

$$\frac{dV}{dt} = -kV$$

Since we know that at $t = 0$, we have $V = 50$ and $V' = -10$, we can determine the value of $k$:

$$-10 = -k(50) \implies k = 1/5$$

The equation is separable, so we can solve it in the usual manner:

$$\frac{dV}{dt} = -\frac{V}{5}$$

$$\frac{dV}{V} = -\frac{dt}{5}$$

$$\int \frac{dV}{V} = -\int \frac{dt}{5}$$

$$\ln |V| = -\frac{t}{5} + c$$

$$|V| = e^{c}e^{-t/5}$$

$$V = (\pm e^{c})e^{-t/5}$$

$$V = Ae^{-t/5}$$

At $t = 0$, we have $V = 50$, which gives us $50 = Ae^{0} = A$. So, we have

$$V = 50e^{-t/5}$$
(b) Suppose that in addition to the water leaking out as in part (a), the cook is also adding salt to the (initially pure) water at a constant rate of 10g per minute.

Find the expression for the concentration of salt in the water as a function of time.

(You may assume for this problem that the salt mixes thoroughly and instantly once it is added to the pot. Also, you may assume that the salt occupies no volume; so, the expression for volume that you found in part (a) is unaffected.)

**Solution:** Instead of attempting to write down a differential equation for the concentration of salt, \( c(t) \), we will write down an equation for the quantity of salt, \( q(t) \). The “rate in” is just the rate at which the cook is adding salt, which is 10; the “rate out” is the product of the rate at which water is leaking out and the concentration of salt in that water.

\[
\frac{dq}{dt} = \text{(rate in)} - \text{(rate out)} = \left(10\right) - \left(c(t) \left| \frac{dV}{dt} \right| \right) = 10 + \left( \frac{q}{V} \right) \left( \frac{dV}{dt} \right)
\]

Since we know that the equations for volume are unaffected by the introduction of the salt into the system, we know that our previous formulas for \( V \) still hold. So our equation becomes

\[
\frac{dq}{dt} = 10 + \left( \frac{q}{50e^{-t/5}} \right) (50e^{-t/5})' = 10 - q/5
\]

So, our differential equation for \( q \) is just

\[
\frac{dq}{dt} = 10 - q/5 = -\frac{1}{5} (q - 50)
\]

Again, this is separable.

\[
\frac{dq}{q - 50} = -\frac{1}{5} \, dt
\]

\[
\int \frac{dq}{q - 50} = -\int \frac{1}{5} \, dt
\]

\[
\ln |q - 50| = -\frac{1}{5} t + c
\]

\[
|q - 50| = e^c e^{-\frac{1}{5}t}
\]

\[
q - 50 = (\pm e^c) e^{-\frac{1}{5}t}
\]

\[
q - 50 = Ae^{-\frac{1}{5}t}
\]

Of course we know that at \( t = 0 \), there is no salt in the water, so \( q = 0 \); so \( 0 - 50 = Ae^0 \), so \( A = -50 \). We can then solve for \( q(t) \) as

\[
q(t) = 50(1 - e^{-\frac{1}{5}t})
\]

and so

\[
c(t) = \frac{q(t)}{V(t)} = \frac{50(1 - e^{-\frac{1}{5}t})}{50e^{-t/5}} = e^{t/5} - 1
\]
5. Show that the solution to the initial value problem
\[ y' + e^{-x} + (y - e^{-x})(x^4 + y^2) = 0, \quad y(0) = 2 \]
is positive and decreasing for all values of \( x \). (Hint: What can you say about the curve \( y = e^{-x} \)?)

**Solution:** As suggested by the hint, we consider the function \( y = e^{-x} \), and observe it is a solution, since
\[ (e^{-x})' + e^{-x} + (e^{-x} - e^{-x})(x^4 + y^2) = 0 \]
for all values of \( x \).

The differential equation is a slope field differential equation since we can solve for \( y' \) in terms of \( x \) and \( y \); so, solutions cannot cross.

We also notice that our initial condition (the point \((0, 2)\)) lies above the curve \( y = e^{-x} \).

Combining these observations, we see that since the solution to the given initial value problem begins above the curve \( y = e^{-x} \), it must remain always above that curve.

Of course \( y = e^{-x} \) is always positive, and so the solution to our initial value problem must be also.

To see that the solution is always decreasing, we merely point out that in the differential equation, all of the terms other than \( y' \) are always positive.

The first other term \((e^{-x})\) is an exponential and thus positive.

The second other term is the product of \((y - e^{-x})\), which is positive since we’ve already shown that \( y \) will always be greater than \( e^{-x} \); and \((x^4 + y^2)\), which is positive since it is the sum of two even powers.

Since the sum of \( y' \) and these positive terms must equal 0, we conclude that \( y' \) must be negative. Therefore, the solution to the initial value problem must be decreasing.
**Bonus Question:**

Find the *surface area* of the torus (doughnut) obtained by rotating the circle \( x^2 + y^2 = r^2 \) around the line \( x = R \). (Assume \( R > r \))

**Solution 1 (of 3):** In the same way that rotating an area creates volume, rotating a length creates area. In particular, rotating a tiny piece of length \( ds \) around a circle of circumference \( C \) creates an area \( Cds \).

As shown in the picture below, if we slice up the circle horizontally, each value of \( y \) corresponds to two pieces of length \( ds \). Each of these is rotated around a different circle.

![Diagram of a torus and its cross-section](image)

The area can then be written as an integral, where there are two terms – one corresponding to the right piece, and one to the left piece.

\[
\int_{-r}^{r} \left( (2\pi(R - x) \, ds) + (2\pi(R + x) \, ds) \right)
\]

Of course we get some nice cancellation, and the integral becomes

\[
\int_{-r}^{r} 2\pi(2R) \, ds = 2\pi R \int_{-r}^{r} 2 \, ds
\]

Of course the remaining integral is just the sum of the lengths of all of the little pieces making up the circle, and so it equals the circumference of the circle. Thus the expression simplifies to
\[2\pi R(2\pi r) = 4\pi^2 R r\]
Solution 2: Recall that the surface area of the sphere is the derivative with respect to radius of the volume of the sphere:

\[ A = 4\pi r^2 = \frac{d}{dr} \left( \frac{4}{3}\pi r^3 \right) = \frac{d}{dr} (V) \]

In fact, the same is true for the torus.

Recall that the volume of a torus of radii \( r \) and \( R \) (which we will write as \( T_{r,R} \)) is

\[ V(T_{r,R}) = 2\pi^2 R r^2 \]

Let us write down the limit definition of the derivative of this expression, with respect to \( r \):

\[ \frac{d}{dr} V(T_{r,R}) = \lim_{h \to 0} \frac{V(T_{r+h,R}) - V(T_{r,R})}{h} \]

Of course as \( h \) approaches 0, the difference in the numerator can be seen (geometrically) to be just the volume of a thin shell of thickness \( h \) around the surface of the torus \( T_{r,R} \). So, we can write that volume simply as the area multiplied by the thickness:

\[ \frac{d}{dr} V(T_{r,R}) = \lim_{h \to 0} \frac{A(T_{r,R}) \cdot h}{h} \]

The \( h \)'s cancel, and the limit is then easily seen to just be the area.

So, we can interpret this to mean that the area of the torus can be computed as the derivative of the volume with respect to \( r \). So, we get

\[ A(T_{r,R}) = \frac{d}{dr} V(T_{r,R}) = \frac{d}{dr} 2\pi^2 R r^2 = 4\pi^2 R r \]

Solution 3: In class, we stated Pappas’ Theorem for volumes – which states that the volume created by a rotated area is just the area times the distance travelled by the centroid of the area.

In fact there is a second Pappas’ Theorem, which makes a similar statement for areas. Namely, the area created by a rotated piece of length is just the length times the distance travelled by the centroid of the length.

Of course, the centroid of a circle is just the center of the circle.

So, using this second version of the theorem, we conclude that the area of the torus is

\[ A = (2\pi r)(2\pi R) = 4\pi^2 R r \]