EXAM 1 SOLUTIONS
You have 2 hours.
No notes, no books.
YOU MUST SHOW ALL WORK TO RECEIVE CREDIT
Good luck!

Name ________________________________

1. ___________ (/10 points)

2. ___________ (/15 points)

3. ___________ (/15 points)

4. ___________ (/15 points)

5. ___________ (/15 points)

6. ___________ (/15 points)

7. ___________ (/15 points)

Bonus: ___________ (/10 points)

Total ___________ (/100 points)
1. Simplify the following (Hint – the answer is an integer):

\[ \log_2(3^2) \log_3(4^2) \cdots \log_7(8^2) \]

**Solution:**

Using the laws of logarithms, we rewrite this first as

\[ 2 \log_2(3) 2 \log_3(4) \cdots 2 \log_7(8) \]

and then as

\[ 2^6 \frac{\ln 3 \ln 4}{\ln 2 \ln 3} \cdots \frac{\ln 8}{\ln 7} \]

There is much cancellation, leaving only

\[ \frac{2^6 \ln 8}{\ln 2} = 2^6 \log_2 8 = 2^6 \cdot 3 = 192 \]
2. Use the $\epsilon - \delta$ definition of a limit to prove that

$$\lim_{x \to 0} x^5 = 0$$

**Solution:**

**Thinking:** We know that, starting from an inequality of the form $0 < |x - 0| < \delta$, we will eventually need to conclude that

$$|x^5 - 0| < \epsilon$$

Of course this is equivalent to $|x| < \epsilon^{1/5}$; which suggests to us that our choice of $\delta$ should be $\delta = \epsilon^{1/5}$.

We are now ready to begin our proof.

**Proof:**

Let $\epsilon > 0$ be given, and choose $\delta = \epsilon^{1/5}$. The reasoning below shows that this choice satisfies the required condition:

Assume $0 < |x - 0| < \delta$. Then $|x| < \epsilon^{1/5}$, which implies $|x^5| < \epsilon$, or $|x^5 - 0| < \epsilon$, which means that $|f(x) - L| < \epsilon$.

So the chosen $\delta$ satisfies the required condition, and thus 0 is the limit.
3. Find the following limits, using any methods we have discussed in class. (You must justify your results!)

a) \[ \lim_{x \to 2} x^2 e^{x^2+1} \]

**Solution:**
We know that \( x^2 + 1 \) is continuous since it is a polynomial, and \( e^x \) is continuous since it is an exponential. Therefore, by the continuity theorems, we know that their composition \( e^{x^2+1} \) is continuous. Furthermore, \( x^2 \) is continuous since it is a polynomial. So we conclude that the product \( x^2 e^{x^2+1} \) is continuous, which allows us to conclude that

\[ \lim_{x \to 2} x^2 e^{x^2+1} = 2^2 e^{2^2+1} = 4 e^5 \]

b) \[ \lim_{x \to 0} x^3 \cos \left( \frac{\pi}{x} \right) \sin(x^2) \]

**Solution:**
This function is not continuous at \( x = 0 \), because in particular, it is not defined there. We use the Squeeze Theorem, as follows:

Note that \( -1 < \cos \left( \frac{\pi}{x} \right) \) and \( -1 < \sin(x^2) < 1 \); therefore, we also have that \( -1 < \cos \left( \frac{\pi}{x} \right) \sin(x^2) < 1 \). Multiplying this inequality by \( x^3 \) gives us

\[ -|x^3| < x^3 \cos \left( \frac{\pi}{x} \right) \sin(x^2) < |x^3| \]

(Note that the absolute values around the \( x^3 \)'s are necessary, since we don’t know if the \( x^3 \) that we multiplied by is positive or negative, so the directions of the inequalities may have shifted...)

We know that \( \lim_{x \to 0} |x^3| = 0 \), as is \( \lim_{x \to 0} -|x^3| = 0 \). Therefore, by the Squeeze Theorem, we conclude that the desired limit is also zero.
4. Find the value of $c$ for which the following function is continuous:

$$f(x) = \begin{cases} \frac{x^2 + x - 6}{x - 2} & \text{if } x < 2 \\ cx & \text{if } x \geq 2 \end{cases}$$

**Solution:**

For this function to be continuous, we need the left hand and right hand limits to both exist and be equal.

The left hand limit is

$$\lim_{x \to 2^-} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2^-} \frac{(x + 3)(x - 2)}{x - 2} = \lim_{x \to 2^-} (x + 3) = 5$$

The right hand limit is

$$\lim_{x \to 2^+} cx = 2c$$

So, for the above function to be continuous, we need to have $5 = 2c$, or $c = \frac{5}{2}$.
5. Use the limit definition of a derivative to find the derivative of the following function:

\[ f(x) = x^3 - 2x^2 \]

**Solution:**

By definition,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x + h)^3 - 2(x + h)^2 - (x^3 - 2x^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - 2(x^2 + 2xh + h^2) - (x^3 - 2x^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{(3x^2h + 3xh^2 + h^3) - 2(2xh + h^2)}{h}
\]

\[
= \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) - 2(2x + h)
\]

\[
= 3x^2 - 4x
\]
6. Use the Taylor polynomial of degree 4 for \( f(x) = e^x \) at \( a = 0 \) to find a rational estimate for \( e \) itself. You may use the fact that \((e^x)' = e^x\).

Express your answer as the ratio of two integers, in reduced form.

**Solution:**

The general Taylor polynomial of degree four is given by

\[
T(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \frac{f^{[4]}(a)}{4!}(x - a)^4
\]

Noting that \( f^{[n]}(x) = e^x \) for all values of \( n \), we see that \( f^{[n]}(0) = e^0 = 1 \) for all \( n \). This allows us to write down the Taylor polynomial of degree four for \( e^x \) as

\[
T(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4
\]

We want to estimate \( e \) itself, which of course is equal to \( e^1 = f(1) \). So our approximation of \( e \) is given by \( T(1) \), which is

\[
T(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24}
\]
7. Use the information from the graph of $f(x)$ below to prove that, based on the $\epsilon - \delta$ definition of a limit, 5 is NOT the limit of $f(x)$ as $x$ approaches 6.

(You may assume that for $x < 6$, we have $f(x) < 5$, and for $x > 6$, we have $f(x) > 7$)

(Hint – you need only show that there is an $\epsilon$ for which there is no $\delta$ that satisfies the usual condition, and explain why that condition is not satisfied.)

![Graph of f(x)](image)

**Solution:**

Choose $\epsilon = 1$, and let $\delta$ be arbitrarily given.

There will always be some value of $x$ greater than 6 that satisfies the hypothesis

$$0 < |x - 6| < \delta$$

But since this value of $x$ is greater than six, we know that $f(x) > 7$, which implies that $f(x) - 5 > 2$, and thus $|f(x) - 5| > 2$. Therefore we can certainly conclude that

$$|f(x) - 5| \not< 1$$

This shows that the chosen $\delta$ does not satisfy the required condition

$$0 < |x - 6| < \delta \implies |f(x) - 5| < 1$$

Since $\delta$ was chosen arbitrarily, we conclude that no $\delta$ can satisfy the required condition, so 5 cannot be the limit.
**Bonus Question:**

Use the $\epsilon - \delta$ definition of a limit to show that if

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = K$$

then

$$\lim_{x \to a} (f(x) + g(x)) = L + K$$

**Solution:**

**Thinking:** We know that we will eventually need to conclude

$$|f(x) + g(x) - (L + K)| < \epsilon$$

To try to accomplish this, note that

$$|f(x) + g(x) - (L + K)| < |f(x) - L| + |g(x) - K|$$

If we could somehow ensure that each of the items on the right side of the above equation were less than $\epsilon/2$, then certainly we could conclude what we want.

Of course, we are given that $L$ is the limit of $f$, so we know that there must exist some $\delta_1$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \epsilon/2$$

We are also given that $K$ is the limit of $g$, so we know that there must exist some $\delta_2$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - K| < \epsilon/2$$

To ensure that both of these hold, we decide to choose $\delta = \min\{\delta_1, \delta_2\}$, since

$$0 < |x - a| < \min\{\delta_1, \delta_2\} \implies 0 < |x - a| < \delta_1$$

and

$$0 < |x - a| < \min\{\delta_1, \delta_2\} \implies 0 < |x - a| < \delta_2$$

(next page...)

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Proof:

Let \( \varepsilon > 0 \) be given, and choose \( \delta \) as indicated in the “thinking” section above. We must show that this choice satisfies the usual condition.

We know from our choice of \( \delta \) that

\[
0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon/2
\]

and

\[
0 < |x - a| < \delta \implies 0 < |x - a| < \delta_2 \implies |g(x) - K| < \varepsilon/2
\]

Therefore,

\[
|f(x) + g(x) - (L + K)| < |f(x) - L| + |g(x) - K| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

as desired.