EXAM II SOLUTIONS
Math 41, Fall 2001.
You have 2 hours.
No notes, no books, no calculators.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT
Good luck!

Name _______________________________________
ID number____________________________

1. __________ ( /20 points)  “On my honor, I have neither given nor
received any aid on this examination. I
have furthermore abided by all other
aspects of the honor code with respect to
this examination.”

2. __________ ( /30 points)

Signature: _____________________________

3. __________ ( /20 points)  Circle your TA’s name:

4. __________ ( /30 points)  Ted Hwa (2 and 7)

5. __________ ( /20 points)  Yu Yan (3 and 8)

6. __________ ( /30 points)  Brett Parker (4 and 9)

   Ryan Vinroot (5 and 10)
   Heaseung Kwon (A02)
   Alex Meadows (A03)

Bonus ___________ ( /15 points)  Circle your section meeting time:

Total ___________ ( /150 points)  11:00am  1:15pm  7pm
1. Find the derivatives of the following functions:

a) \[ f(x) = \ln(1 + e^{\sin x}) \]

Solution:
\[
\begin{align*}
\frac{d}{dx} f(x) &= \frac{1}{1 + e^{\sin x}} (1 + e^{\sin x})' \\
&= \frac{1}{1 + e^{\sin x}} (e^{\sin x} (\sin x)') \\
&= \frac{1}{1 + e^{\sin x}} (e^{\sin x} (\cos x)) \\
&= \frac{e^{\sin x} \cos x}{1 + e^{\sin x}}
\end{align*}
\]

b) \[ g(x) = \sqrt{1 + x^x} \quad \text{(assume } x > 0) \]

Solution:
\[
\begin{align*}
g(x) &= (1 + x^x)^{\frac{1}{2}} \\
g'(x) &= \frac{1}{2} (1 + x^x)^{-\frac{1}{2}} (1 + x^x)' \\
&= \frac{(x^x)'}{2\sqrt{1 + x^x}}
\end{align*}
\]

To evaluate this, we need to find the derivative of \( x^x \):
\[
\begin{align*}
y &= x^x \\
\ln y &= x \ln x \\
\frac{y'}{y} &= x \left( \frac{1}{x} \right) + 1 \ln x \\
y' &= y \left( 1 + \ln x \right) = x^x (1 + \ln x)
\end{align*}
\]

Plugging this in to our previous equation, we get
\[
g'(x) = \frac{x^x (1 + \ln x)}{2\sqrt{1 + x^x}}
\]
2. (a) Find the derivative of \( xe^{x^2} \).

Solution:

\[
\left( xe^{x^2} \right)' = (x)'(e^{x^2}) + (x)(e^{x^2})' \\
= e^{x^2} + x(e^{x^2}(2x)) \\
= (1 + 2x^2)e^{x^2}
\]

(b) Find the derivative of \((\sin x)e^{x^2}\)

Solution:

\[
\left( (\sin x)e^{x^2} \right)' = (\sin x)'(e^{x^2}) + (\sin x)(e^{x^2})' \\
= (\cos x)(e^{x^2}) + (\sin x)(e^{x^2}(2x)) \\
= (\cos x + 2x \sin x)(e^{x^2})
\]

(c) Let \( f \) be differentiable; find (in terms of \( f \) and \( f' \)) the derivative of

\( f(x)e^{x^2} \)

Solution:

\[
\left( f(x)e^{x^2} \right)' = (f(x))'(e^{x^2}) + (f(x))(e^{x^2})' \\
= f'(x)(e^{x^2}) + f(x)(e^{x^2}(2x)) \\
= (f'(x) + 2xf(x))(e^{x^2})
\]
3. (a) The “Folium of Descartes” is the set of solutions to the equation

\[ x^3 + y^3 = 3axy \]

where \( a \) is a constant. Find \( \frac{dy}{dx} \) as a function of \( x \) and \( y \).

**Solution:**

\[ x^3 + y^3 = 3axy \]

\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3a(1)(y) + 3ax \frac{dy}{dx} \]

\[ 3y^2 \frac{dy}{dx} - 3ax \frac{dy}{dx} = 3ay - 3x^2 \]

\[ \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \]

(b) Find \( \frac{dy}{dx} \) as a function of \( x \) and \( y \) for the “Kampyle of Eudoxus”, which is the set of solutions to

\[ a^2x^4 = b^4(x^2 + y^2) \]

where \( a \) and \( b \) are constants.

**Solution:**

\[ a^2x^4 = b^4(x^2 + y^2) \]

\[ 4a^2x^3 = 4b^4(x + 2y \frac{dy}{dx}) \]

\[ 4a^2x^3 - 2b^4 x = 2b^4 y \frac{dy}{dx} \]

\[ \frac{dy}{dx} = \frac{2a^2x^3 - b^4 x}{b^4 y} \]
4. (a) Find the first, second, and third derivatives of

\[ f(x) = \ln(1 + x) \]

**Solution:**

\[ f'(x) = \frac{1}{1 + x} = (1 + x)^{-1} \]

\[ f''(x) = (-1)(1 + x)^{-2} = \frac{-1}{(1 + x)^2} \]

\[ f'''(x) = (-1)(-2)(1 + x)^{-3} = \frac{2}{(1 + x)^3} \]

(b) Noticing a pattern in the answers from part (a), what is the \( n \)th derivative of \( f(x) \)?

**Solution:** Every time we take another derivative, the exponent is decreased by one; and, the constant in front becomes a larger factorial, and changes sign. This pattern continues. So,

\[ f^{[n]}(x) = \left( -1 \right) \left( -2 \right) \cdots \left( - (n - 1) \right) \left( 1 + x \right)^{-n} \]

\[ = (-1)^{n-1}(n - 1)!\left( 1 + x \right)^{-n} = \frac{(-1)^{n-1}(n - 1)!}{(1 + x)^n} \]
(c) Using the results from parts (a) and (b), write down the Taylor series (with \( a = 0 \)) for \( f(x) = \ln(1 + x) \). Write your answer in the form

\[
T(x) = (0\text{th term}) + (1\text{st term}) + (2\text{nd term}) + \ldots + (n\text{th term}) + \ldots
\]

**Solution:**

\[
f(a) = f(0) = \ln(1 + 0) = 0
\]
\[
f'(a) = f'(0) = (1 + 0)^{-1} = 1
\]
\[
f''(a) = f''(0) = (-1)(1 + 0)^{-2} = -1
\]
\[
: \quad f^{[n]}(a) = f^{[n]}(0) = \frac{(-1)^{n-1}(n-1)!}{(1 + 0)^n} = (-1)^{n-1}(n-1)!
\]

\[
T(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \ldots + \frac{f^{[n]}(a)(x - a)^n}{n!}
\]
\[
= 0 + x + \frac{(-1)x^2}{2!} + \ldots + \frac{(-1)^{n-1}(n-1)!x^n}{n!}
\]
\[
= 0 + x - \frac{x^2}{2} + \ldots + \frac{(-1)^{n-1}x^n}{n}
\]

(d) Use the second order Taylor polynomial to estimate \( \ln(1.001) \).

**Solution:** The second order Taylor polynomial for \( \ln(1 + x) \) is

\[
T_2(x) = 0 + x - \frac{x^2}{2}
\]

Plugging in \( x = .001 \), we get

\[
\ln(1.001) \approx (.001) - .5(.000001) = .001 - .0000005 = .0009995
\]
5. Find the absolute maximum value of the function

\[ f(x) = x - x^3 \]

defined on the interval \([-3, 2]\). Explain all of your reasoning.

**Solution:**

\[ f'(x) = 1 - 3x^2 \]

Since \( f \) is a continuous function defined on a closed interval, the Extreme Value Theorem tells us that the absolute maximum is attained somewhere in the given interval.

Since \( f' \) is defined everywhere, the only critical numbers, and thus the only candidates for local maxima, are the points where \( f' = 0 \); namely, \( x = \pm \sqrt{1/3} \).

The absolute maximum must be attained either at one of these critical numbers, or at one of the end points. So, we check the value of the function at those four points:

\[
\begin{align*}
  f(-3) &= (-3) - (-3)^3 = 24 \\
  f\left(-\sqrt{1/3}\right) &= \left(-\sqrt{1/3}\right) - \left(-\sqrt{1/3}\right)^3 = -\frac{2}{3}\sqrt{1/3} \quad (< 0) \\
  f\left(\sqrt{1/3}\right) &= \left(\sqrt{1/3}\right) - \left(\sqrt{1/3}\right)^3 = \frac{2}{3}\sqrt{1/3} \quad (< 1) \\
  f(2) &= (2) - (2)^3 = -6 \\
\end{align*}
\]

The greatest of these values is 24, which is achieved at \( x = -3 \).
6. Evaluate the following limits using L’Hôpital’s Rule:

(a)

\[ \lim_{x \to \infty} x^{(1/x)} \]

Solution:

\[
y = \lim_{x \to \infty} x^{(1/x)} \\
\ln y = \ln \lim_{x \to \infty} x^{(1/x)} \\
= \lim_{x \to \infty} \ln x^{(1/x)} \\
= \lim_{x \to \infty} (1/x) \ln x \\
= \lim_{x \to \infty} \frac{\ln x}{x} \quad \left( \frac{\infty}{\infty}; \text{ apply L’Hôpital} \right) \\
= \lim_{x \to \infty} \frac{1/x}{1} \\
\ln y = 0 \\
y = e^0 = 1
\]

(b)

\[ \lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} \]

Solution: Applying L’Hôpital’s rule gives us a series of indeterminate forms of type \( \frac{0}{0} \); after five applications, we finally arrive at a determinable limit:

\[
= \lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{5x^4} \\
= \lim_{x \to 0} \frac{-\sin x + x}{20x^3} \\
= \lim_{x \to 0} \frac{-\cos x + 1}{60x^2} \\
= \lim_{x \to 0} \frac{\sin x}{120x} \\
= \lim_{x \to 0} \frac{\cos x}{120}
\]

This last function is continuous, so we just plug in and conclude that the limit is \(1/120\).
(c) Rewrite the limit from part (b) using the Taylor series for \( \sin x \) (below), and use this as an alternative method to compute the limit.

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
\]

**Solution:** The limit becomes

\[
\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - \left(x - \frac{x^3}{3!}\right)}{x^5}
\]

\[
= \lim_{x \to 0} \frac{\left(\frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}{x^5}
\]

\[
= \lim_{x \to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \ldots\right)
\]

\[
= \frac{1}{5!} + \lim_{x \to 0} x^2 \left(-\frac{1}{7!} + \frac{x^2}{9!} \ldots\right)
\]

\[
= \frac{1}{120} + 0 = \frac{1}{120}
\]
**Bonus Question:**

We are given a tank full of water, which is evaporating out of the top at a rate that is proportional to the surface area of water that is exposed to the air.

However, we are not given any information about the actual shape of the tank.

We define \( V(h) \) to be the volume of water in the tank when the depth of the water is \( h \). We define \( A(h) \) to be the exposed surface area when the depth of the water is \( h \).

For this problem, you may assume also that the rate of change of \( V(h) \) with respect to \( h \) is \( A(h) \).

Use the chain rule to show that the depth of the water decreases at a constant rate (with respect to time) as a result of the evaporation, independent of the shape of the tank.

(Solution on next page...)

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Solution: Volume is a function of height, and height is a function of time. So, by the Chain
rule, we have

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt}$$

We are given that the rate of evaporation (in other words, the rate at which water is leaving
the tank) is proportional to area. But the rate of evaporation is the rate at which water is
leaving the tank, and is thus $\frac{dV}{dt}$. So we conclude

$$\frac{dV}{dt} = -kA(h)$$

for some constant $k$.

We are also given specifically that

$$\frac{dV}{dh} = A(h)$$

Putting these two facts into the Chain rule, we get

$$-kA(h) = A(h) \frac{dh}{dt}$$

$$-k = \frac{dh}{dt}$$

So, the depth of the water ($h$) decreases at a constant rate with respect to time. This result is
independent of the shape of the tank, since we made no assumptions about either $A$ or $V$. 