Homework 6 Solutions

1. (a) It is enough to show that \( \lim_{(x,y) \to (0,0)} |f(x,y)| = 0. \) But, for \( |x| < 1, \)
\[
\frac{|x^5 + x^3y^2|}{x^4 + y^2} = \frac{|x^4 + x^2y^2|}{x^4 + y^2} < |x| \frac{|x^4 + y^2|}{x^4 + y^2} = |x|.
\]
Since \( |x| \to 0 \) as \( (x,y) \to (0,0) \), this shows that \( f(x,y) \) is continuous at \( (0,0) \).

(b) \[
\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^5}{h} = 1.
\]
\[
\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0.
\]

(c) If \( f \) were differentiable, then its derivative would be given by the Jacobian matrix \( J = \begin{bmatrix} 1 & 0 \end{bmatrix} \). But, with \( \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \),
\[
\frac{f((0,0)+\mathbf{v}) - f(0,0) - J\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{a^2 + b^2}} \left( \frac{a^5 + a^3b^2}{a^4 + b^2} - a \right)
\]
\[
= \frac{a^3b^2 - ab^2}{\sqrt{a^2 + b^2}(a^4 + b^2)}.
\]
If \( f \) were differentiable, then the limit of this expression as \( (a,b) \to (0,0) \) would be zero. But if we approach zero along \( a = b > 0 \) we get
\[
\lim_{a \to 0^+} \frac{a^5 - a^3}{\sqrt{2}(a^5 + a^3)} = \lim_{a \to 0^+} \frac{a^2 - 1}{2(a^2 + 1)} = -\frac{1}{\sqrt{2}}.
\]
Thus, \( f \) is not differentiable at \( (0,0) \).