MATH 51
SECOND SAMPLE MIDTERM #2
SOLUTIONS

90 minutes

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. No calculators.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>16</td>
<td></td>
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<td>3.</td>
<td>18</td>
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<td>4.</td>
<td>17</td>
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<td>18</td>
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<tr>
<td>6.</td>
<td>16</td>
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</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
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1. Suppose that \( f(x, y) : \mathbb{R}^2 \to \mathbb{R} \) is differentiable. Consider the unit vectors
\[
\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}.
\]
Assume \( D_{\mathbf{u}_1} f(3, 7) = 4 \), and \( D_{\mathbf{u}_2} f(3, 7) = 6 \). Can you determine \( D_{\mathbf{u}_3} f(3, 7) \)? If so, do it. If not, explain why not.

"Yes." Since the derivative is a linear transformation, if we write \( \mathbf{u}_3 = a\mathbf{u}_1 + b\mathbf{u}_2 \), then \( D_{\mathbf{u}_3} f(3, 7) = aD_{\mathbf{u}_1} f(3, 7) + bD_{\mathbf{u}_2} f(3, 7) \). Since \( \mathbf{u}_1, \mathbf{u}_2 \) span \( \mathbb{R}^2 \), it is possible to find \( a, b \).

To do it, we consider the augmented matrix:
\[
\begin{bmatrix}
1/\sqrt{2} & 3/5 & -2/\sqrt{5} \\
1/\sqrt{2} & 4/5 & -1/\sqrt{5} \\
1/\sqrt{2} & 3/5 & -2/\sqrt{5} \\
0 & 1/5 & 1/\sqrt{5} \\
1/\sqrt{2} & 0 & -5/\sqrt{5} \\
0 & 1/5 & 1/\sqrt{5} \\
1 & 0 & -\sqrt{10} \\
0 & 1 & \sqrt{5}
\end{bmatrix}
\]
Thus,
\[
D_{\mathbf{u}_3} f(3, 7) = -4\sqrt{10} + 6\sqrt{5}.
\]

**ALTERNATE APPROACH:** Use \( \nabla f(3, 7) \cdot \mathbf{u}_1 = 4 \), and \( \nabla f(3, 7) \cdot \mathbf{u}_2 = 6 \) to find that
\[
\nabla f(3, 7) = \begin{bmatrix} 16\sqrt{2} - 30 \\ 30 - 12\sqrt{2} \end{bmatrix}.
\]
Then,
\[
D_{\mathbf{u}_3} f(3, 7) = \nabla f(3, 7) \cdot \mathbf{u}_3
= \frac{-32\sqrt{2}}{\sqrt{5}} + \frac{60}{\sqrt{5}} - \frac{30}{\sqrt{5}} + \frac{12\sqrt{2}}{\sqrt{5}}
= -4\sqrt{10} + 6\sqrt{5}.
\]
2. Suppose that $f$ is a differentiable function from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $f(1, 2) = (5, -6)$ and $Jf_{(1, 2)} = \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix}$.

(a) Estimate $f(a, b)$.

$$(a, b) = (1, 2) + \mathbf{v} \text{ where } \mathbf{v} = \begin{bmatrix} a - 1 \\ b - 2 \end{bmatrix}. \text{ Thus,}$$

$$f(a, b) \approx f(1, 2) + Jf_{(1, 2)} \mathbf{v}$$

$$= (5, -6) + \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} a - 1 \\ b - 2 \end{bmatrix}$$

$$= (5, -6) + \begin{bmatrix} 2a - 2 \\ -3a + 3 - b + 2 \end{bmatrix}$$

$$= (3 + 3a, -1 - 3a - b)$$

(b) Express the matrix $Jf_{(1, 2)}$ in terms of the basis $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Let $M = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$. This matrix changes basis from $\mathbf{w}_1, \mathbf{w}_2$ to the standard basis. $M^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ changes back. Thus, in the basis $\mathbf{w}_1, \mathbf{w}_2$, the matrix is given by

$$M^{-1}Jf_{(1, 2)}M = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 & 15 \\ -12 & -13 \end{bmatrix}$$
3. (a) Suppose that \( f: D \subset \mathbb{R}^2 \to \mathbb{R} \) is continuous at \((0, 0)\) and \(f(0, 0) > 0\). Prove that there exists \( r > 0 \) so that \( f(x, y) > 0 \) for all points \((x, y)\) inside the ball of radius \( r \) around \((0, 0)\).

Let \( a = f(0, 0) > 0 \). Set \( \epsilon = a/2 \). Since \( f \) is continuous at \((0, 0)\), \( \lim_{(x,y) \to (0,0)} f(x, y) = a \). By definition, this means that there exists an \( r > 0 \) so that \( \|(x, y) - (0, 0)\| < r \) implies \(|f(x, y) - a| < \epsilon = a/2\). In other words, \((x, y)\) inside the ball of radius \( r \) implies that \( a/2 < f(x, y) < 3a/2\); in particular, \( f(x, y) > 0 \).

(b) Each of the following functions is discontinuous at \((0, 0)\). For each one, decide if the discontinuity is removable. (Show your work!)

\[
\begin{align*}
\text{(i)} & \quad g(x, y) = \frac{x^2}{x^2 + y^2}, \\
\text{(ii)} & \quad h(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}}.
\end{align*}
\]

(i) \( \lim_{(x,y) \to (0,0)} g(x, y) \) does not exist so \( g \) has a non-removable discontinuity at \((0, 0)\). To see this, observe that \( x = 0 \) implies \( g \) is zero. However, when \( y = 0, g = \frac{x^2}{x^2} = 1 \). Thus, approaching \((0, 0)\) from different directions gives different limits.
(ii) Observe that

\[
0 \leq \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} \leq \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \sqrt{x^2 + y^2} = 0
\]

Therefore, \( \lim_{(x,y) \to (0,0)} f(x, y) = 0 \). Since this limit exists, \( f \) has a removable discontinuity at \((0, 0)\).
4. A dog, Fido, is tied to two leashes, each 5 feet long. The leashes are tied to stakes in the ground, one at position (2, 0) and the other at (−2, 0). Suppose that the temperature (in degrees Celsius) at a point (x, y) is given by \( T(x, y) = 3x^2 + y^2 \) within the region where Fido can move.

(a) Suppose that Fido is currently at the point (1, 2). In what direction should he move to increase his temperature as quickly as possible. (Give a unit vector.)

Fido should move in the direction of \( \frac{1}{\| \nabla T(1, 2) \|} \nabla T(1, 2) \). We have
\[
\nabla T(x, y) = \begin{bmatrix} 6x \\ 2y \end{bmatrix}, \quad \nabla T(1, 2) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.
\]
Thus, he should move in the direction of
\[
\frac{1}{\sqrt{36 + 16}} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \frac{1}{2\sqrt{13}} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}.
\]
(b) To what point should Fido move to be as warm as possible?

We must maximize \( T(x, y) \) on the region bounded by the curves \((x-2)^2 + y^2 = 25\) for \(-3 \leq x \leq 0\), and \((x+2)^2 + y^2 = 25\) for \(0 \leq x \leq 3\). Since \( \nabla T(x, y) = \begin{bmatrix} 6x \\ 2y \end{bmatrix} \), the only critical point in the interior is \((0, 0)\).

Restricted to the boundary, \( T \) takes the form
\[
T(x) = \begin{cases} 
3x^2 + 25 - (x - 2)^2 & -3 \leq x \leq 0 \\
3x^2 + 25 - (x + 2)^2 & 0 < x \leq 3.
\end{cases}
\]
Then, \( T(x) \) is not differentiable when \( x = 0 \). Otherwise,
\[
T'(x) = \begin{cases} 
4x + 4 & -3 \leq x < 0 \\
4x - 4 & 0 < x \leq 3.
\end{cases}
\]
The critical points of \( T(x) \) are therefore \( x = 0, 1, -1 \). We must also include the end-points, \( x = 3, -3 \).

Now, it is easy to see that when \( x = 0, y = \pm\sqrt{21} \), when \( x = \pm 1, y = \pm 4 \), and when \( x = \pm 3, y = 0 \). To find where \( T(x, y) \) is maximized on this region, it only remains to compare values: \( f(0, 0) = 0, f(0, \pm\sqrt{21}) = 21, f(\pm 1, \pm 4) = 19, \) and \( f(\pm 3, 0) = 27 \). Therefore, the temperature is largest at \((-3, 0)\) and \((3, 0)\).

NOTE: an alternative approach is to use Lagrange multipliers to study \( T \) restricted to the boundary.
5. Match each of the following functions with the picture of its vector field on the region $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. (Note that some of the vector fields are not defined at $(0,0)$.)

(a) $f(x, y) = \left(\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}\right)$  

(b) $f(x, y) = \left(\frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}\right)$

(c) $f(x, y) = (-y, \frac{1}{3}x)$

(d) $f(x, y) = \left(\frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}\right)$

(e) $f(x, y) = (-\frac{1}{3}y, x)$

(f) $f(x, y) = (-x, y)$

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(f) $f(x, y) = (-x, y)$
6. (a) Let \( f(x, y) : D \subset \mathbb{R}^2 \to \mathbb{R} \) be differentiable in its whole domain \( D \). Let \( p(t) = (u(t), v(t)), 0 \leq t \leq 1 \), be a trajectory (i.e. a parametric curve) in \( D \). For each \( t \), let \( \mathbf{v}(t) \) be the vector \[
\begin{bmatrix}
u'(t) \\
v'(t)
\end{bmatrix}.
\]
Assume that, for each \( t \), the directional derivative of \( f \) at the point \( p(t) \) in the direction \( \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||} \) is zero. Show that \( f \) is constant along the trajectory.

The given condition is equivalent to
\[
J_{p(t)} f \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = 0.
\]
Let \( g(t) = f(p(t)) \). Then, by the chain rule and the previous formula,
\[
\frac{dg}{dt} = J_{p(t)} f \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = 0.
\]
It follows that \( g(t) \) is a constant function. In other words, \( f \) takes on the same values at all of the points \( p(t) \).

(b) Suppose that \( f(x, y) \) is as above with \( D \) the set \( x^2 + y^2 < 5 \). Suppose that \( Jf = 0 \) everywhere on this domain. Prove that \( f \) is constant.

We will prove that \( f(a, b) = f(0, 0) \) for all points \((0, 0)\) in the domain. Define \( p(t) = (at, bt) \) for \( 0 \leq t \leq 1 \). Since \( Jf = 0 \), every directional derivative is zero. Thus, by part (a), \( f(p(t)) \) is the same for every \( t \). In particular,
\[
f(a, b) = f(p(1)) = f(p(0)) = f(0, 0).
\]