MATH 51
SECOND SAMPLE MIDTERM #1
SOLUTIONS

90 minutes

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. No calculators.

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1. (a) Let \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) and \( \mathbf{w}_1, \ldots, \mathbf{w}_m \) be two systems of vectors that are orthogonal to each other; i.e. \( \mathbf{v}_i \cdot \mathbf{w}_j = 0 \) for all \( i, j \). Suppose that \( \mathbf{u} \) is a vector in the span of \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) and also in the span of \( \mathbf{w}_1, \ldots, \mathbf{w}_m \). Prove that \( \mathbf{u} = 0 \).

To show that \( \mathbf{u} = 0 \), it is enough to show that \( \mathbf{u} \) has length zero, i.e. \( \mathbf{u} \cdot \mathbf{u} = 0 \). Now, we know that

\[
\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_k \mathbf{v}_k
\]

and

\[
\mathbf{u} = b_1 \mathbf{w}_1 + b_2 \mathbf{w}_2 + \cdots + b_m \mathbf{w}_m.
\]

Thus,

\[
\mathbf{u} \cdot \mathbf{u} = (a_1 \mathbf{v}_1 + \cdots + a_k \mathbf{v}_k) \cdot \mathbf{u} = a_1 \mathbf{v}_1 \cdot (b_1 \mathbf{w}_1 + \cdots + b_m \mathbf{w}_m) + \cdots + a_k \mathbf{v}_k \cdot (b_1 \mathbf{w}_1 + \cdots + b_m \mathbf{w}_m)
\]

\[
= \sum_{i,j} a_i b_j \mathbf{v}_i \cdot \mathbf{w}_j
\]

\[
= 0
\]

(b) Suppose that \( \mathbf{p}, \mathbf{q} \) span a two-dimensional space \( \mathcal{V} \). Let

\[
\bar{\mathbf{q}} = \mathbf{q} - \frac{\mathbf{p} \cdot \mathbf{q}}{||\mathbf{p}||^2} \mathbf{p}.
\]

Prove: (i) \( \mathbf{p} \) and \( \bar{\mathbf{q}} \) are perpendicular, and

(ii) \( \mathbf{p} \) and \( \bar{\mathbf{q}} \) form a basis for \( \mathcal{V} \).

(i) \( \mathbf{p} \cdot \bar{\mathbf{q}} = \mathbf{p} \cdot \mathbf{q} - \frac{\mathbf{p} \cdot \mathbf{q}}{||\mathbf{p}||^2} \mathbf{p} \cdot \mathbf{p} \). But, \( \mathbf{p} \cdot \mathbf{p} = ||\mathbf{p}||^2 \). Thus,

\[
\mathbf{p} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{q} = 0.
\]

(ii) Since \( \mathbf{p}, \bar{\mathbf{q}} \) are 2 vectors in a 2-dimensional space, \( \mathcal{V} \), to show that they form a basis it is enough to show either that they are linearly independent, or that they span \( \mathcal{V} \).

SOLUTION 1. Since \( \mathbf{p}, \mathbf{q} \) are linearly independent, \( \mathbf{p} \) and \( \bar{\mathbf{q}} \) are non-zero. But non-zero, perpendicular vectors are independent. (Proof: if \( a \mathbf{p} + b \bar{\mathbf{q}} = 0 \), then \( a \mathbf{p} \cdot \mathbf{p} + b \mathbf{p} \cdot \bar{\mathbf{q}} = 0 \). Since \( \mathbf{p} \cdot \bar{\mathbf{q}} = 0 \) and \( \mathbf{p} \cdot \mathbf{p} \neq 0 \), this implies \( a = 0 \). Similarly, \( b = 0 \).)

SOLUTION 2. To show that \( \mathbf{p}, \bar{\mathbf{q}} \) span \( \mathcal{V} \), it is enough to check that \( \mathbf{p} \) and \( \bar{\mathbf{q}} \) are in the span. But \( \mathbf{p} = 1 \mathbf{p} + 0 \bar{\mathbf{q}} \) and \( \bar{\mathbf{q}} = \frac{\mathbf{p} \cdot \mathbf{q}}{||\mathbf{p}||^2} \mathbf{p} + \bar{\mathbf{q}} \).
2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly independent vectors in $\mathbb{R}^3$. Let $A$ be a $3 \times 3$ matrix so that

$$Au = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad Av = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad Aw = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}. $$

(a) What is the dimension of the null space of $A$? Explain how you know.

It is easy to see that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ span a 2-dimensional space. Since these vectors are in the column space of $A$, the column space of $A$ is at least 2-dimensional. Equivalently, the null space of $A$ is at most 1-dimensional. Next, notice that

$$A(2u + 2v - w) = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$

Moreover, because $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, $2\mathbf{u} + 2\mathbf{v} - \mathbf{w} \neq \mathbf{0}$. This means that the null space of $A$ is at least one-dimensional. All together, this means that the null space is exactly one-dimensional.

(b) Does $Ax = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ have any solutions? If it does, what is the dimension of the solution set? Also, answer the same questions for $Ax = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

In general, $Ax = b$ either has no solutions or else its solution set is a translate of the null space. In this case the null space is a line. Thus, either there are no solutions or there is a one-dimensional set of solutions.

It follows from part (a) that the column space of $A$ is spanned by $Au$ and $Av$. It is easy to see that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in this span but $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ is not. Thus, $Ax = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has a one-dimensional solution set and $Ax = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ has no solutions.
3. Let \( M = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 2 & 0 \\ 3 & 6 & -1 \end{bmatrix} \).

(a) Find bases for the null space and the column space of \( M \).

Doing Gaussian elimination gives

\[
\begin{bmatrix}
1 & 2 & -1 \\
0 & 0 & 1 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Since the pivots are in the first and third columns, the vectors \(
\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}
\) and \(
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\) form a basis for the column space. Also, from the form of the reduced matrix, it is easy to see that the null space is the line spanned by \(
\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}
\).

(b) Write an equation (or a system of equations) for the space of vectors \( \mathbf{b} \) such that the equation \( M\mathbf{x} = \mathbf{b} \) has at least one solution. What is the connection between this problem and part (a)?

The space of vectors \( \mathbf{b} \) such that the equation \( M\mathbf{x} = \mathbf{b} \) has at least one solution is exactly the column space of \( M \). Thus, we must find an equation for the plane spanned by \(
\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}
\) and \(
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\). For this, it is enough to find a normal vector. We do Gaussian elimination on

\[
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
\end{bmatrix}
\]

It follows that \(
\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\) is perpendicular to the column space. Thus,

\[ x - 2y + z = 0 \]

is an equation for the column space.
4. Suppose that

\[
\begin{align*}
M \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
M \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
M \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

(a) Find \( M^{-1} \).

\[
M^{-1} = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 5 & 5 \\ 3 & 2 & 1 \end{bmatrix}
\]

(b) Find \( M \).

Using Gaussian elimination

\[
\begin{align*}
&\left[ \begin{array}{ccc|ccc}
1 & 4 & 4 & 1 & 0 & 0 \\
1 & 5 & 5 & 0 & 1 & 0 \\
3 & 2 & 1 & 0 & 0 & 1 \\
\end{array} \right] \\
&\sim \left[ \begin{array}{ccc|ccc}
1 & 4 & 4 & 1 & 0 & 0 \\
0 & 1 & 1 & -1 & 1 & 0 \\
0 & -10 & -11 & -3 & 0 & 1 \\
\end{array} \right] \\
&\sim \left[ \begin{array}{ccc|ccc}
1 & 0 & 0 & 5 & -4 & 0 \\
0 & 1 & 1 & -1 & 1 & 0 \\
0 & 0 & -1 & -13 & 10 & 1 \\
\end{array} \right] \\
&\sim \left[ \begin{array}{ccc|ccc}
1 & 0 & 0 & 5 & -4 & 0 \\
0 & 1 & 0 & -14 & 11 & 1 \\
0 & 0 & 1 & 13 & -10 & 1 \\
\end{array} \right]
\end{align*}
\]

Thus,

\[
M = \begin{bmatrix} 5 & -4 & 0 \\ -14 & 11 & 1 \\ 13 & -10 & 1 \end{bmatrix}
\]

(c) Suppose that \( M \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \). What do \( a \), \( b \) and \( c \) represent?

\( a, b \) and \( c \) are the coefficients when \( \mathbf{v} \) is written in the basis

\[
\begin{bmatrix} 1 \\ 1 \\ 3 \\ \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}
\]
5. Suppose that $A$ is a $4 \times 3$ matrix so that $Ax = 0$ has a unique solution. Also, $B$ is a $3 \times 4$ matrix with linearly independent rows, and $C$ is a $3 \times 3$ matrix with a two-dimensional column space.

For each of the following, answer with either a number, “yes,” “no,” or “undetermined.” “Undetermined” means that there is not enough information given to figure it out.

| (a) dimension of the column space of $A = (#/und.)$ | 3 |
| (b) $Ax = b$ is always solvable (yes/no/und.) | NO |
| (c) $Ax = b$ has at most one solution (yes/no/und.) | YES |
| (d) $A$ has linearly independent columns (yes/no/und.) | YES |
| (e) dimension of the null space of $B = (#/und.)$ | 1 |
| (f) dimension of the column space of $AC = (#/und.)$ | 2 |
| (g) dimension of the column space of $BA = (#/und.)$ | Und. |
| (h) dimension of the null space of $AB = (#/und.)$ | 1 |
| (i) $Bx = b$ is always solvable (yes/no/und.) | YES |
| (j) $Cx = b$ has at most one solution (yes/no/und.) | NO |
6. Is it possible to have an invertible linear transformation \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which transforms the line \( y = 0 \) into the line \( x + 3y = 0 \), the line \( x = 0 \) into the line \( x + y = 0 \), and the line \( 2x - y = 0 \) into the line \( x = 0 \)? If so, give a matrix that does it. If not, explain why not.

Transforming the line \( y = 0 \) to the line \( x + 3y = 0 \) is equivalent to transforming the vector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) to a multiple of \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \), say \( T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = a \begin{bmatrix} 3 \\ -1 \end{bmatrix} \). Similarly, in order to transform the line \( x = 0 \) into the line \( x + y = 0 \), we must have \( T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = b \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Thus, \( T \) must be represented by a matrix of the form

\[
M = \begin{bmatrix} 3a & b \\ -a & -b \end{bmatrix}.
\]

Our task now is to select \( a \) and \( b \) so that the line \( 2x - 1 = 0 \) is transformed into the line \( x = 0 \). In other words, we must have \( M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) for some constant \( c \). This leads to the equations

\[
3a + 2b = 0 \\
-a - 2b = c
\]

These are equivalent to \( a = c/2 \) and \( b = -3c/4 \). To make things simple, let’s take \( c = 4 \). Then \( a = 2 \) and \( b = -3 \). We get that the matrix

\[
M = \begin{bmatrix} 6 & -3 \\ -2 & 3 \end{bmatrix}
\]

transforms the lines as required.