MATH 51
SOLUTIONS TO SECOND SAMPLE FINAL EXAM

Three Hours

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. No calculators.

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1. (a) Find bases for the null space and column space of
\[
\begin{bmatrix}
-1 & 2 & -1 & 1 \\
2 & -4 & 1 & 1 \\
0 & 0 & 1 & -3 \\
1 & -2 & 2 & -4
\end{bmatrix}.
\]

Performing Gaussian elimination on the given matrix yields
\[
\begin{bmatrix}
1 & -2 & 0 & 2 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Since the pivots are in the first and third columns, \[
\begin{bmatrix}
-1 \\
2 \\
0 \\
1
\end{bmatrix}
\]
and \[
\begin{bmatrix}
-1 \\
1 \\
1 \\
2
\end{bmatrix}
\]
for a basis for the column space.

Also, \[
\begin{bmatrix}
2 \\
0 \\
3 \\
1
\end{bmatrix}
\]
and \[
\begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix}
\]
form a basis for the null space.

(b) Find an equation (or system of equations) whose solutions are the span of
\[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
4 \\
7
\end{bmatrix}.
\]

We must find vectors perpendicular to all three of these vectors; these will be the normal vectors for the hyperplanes determined by our equations. Performing Gaussian elimination on the matrix with these vectors as ROWS yields
\[
\begin{bmatrix}
1 & 0 & 1 & -2 \\
0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
The null space of this matrix is spanned by
\[
\begin{bmatrix}
2 \\
-3 \\
0 \\
1
\end{bmatrix}
\]
and \[
\begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix}
\]. Thus, a system of equations whose solutions is the given space is
\[
\begin{align*}
2x_1 - 3x_2 + x_4 &= 0 \\
-x_1 - x_2 + x_3 &= 0
\end{align*}
\]
2. (a) Write a matrix that maps $\mathbb{R}^2$ to $\mathbb{R}^2$ and takes \[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\] to \[
\begin{bmatrix}
4 \\
3
\end{bmatrix}
\] and takes \[
\begin{bmatrix}
-2 \\
3
\end{bmatrix}
\] to \[
\begin{bmatrix}
7 \\
-2
\end{bmatrix}
\].

Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ is the matrix which expresses the standard basis in terms of \[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\] and \[
\begin{bmatrix}
-2 \\
3
\end{bmatrix}
\]. Therefore, the matrix that represents our transformation (relative to the standard basis) is

\[
\begin{bmatrix}
4 & 7 \\
3 & -2
\end{bmatrix}
\begin{bmatrix}
3 & 2 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
19 & 15 \\
7 & 4
\end{bmatrix}.
\]

(b) Let $M = \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$. Express $M$ in terms of the basis $v_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$. To express $M$ in the new basis we must compute

\[
A^{-1}MA = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}
\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}
\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}
= \begin{bmatrix} -9 & -8 \\ 14 & 13 \end{bmatrix}.
3. Let $f(x, y) = x^3 + y^3 + 3xy$.
(a) Find the maximum and minimum values of $f$ on the square region $-2 \leq x \leq 1$, $-2 \leq y \leq 1$.

Since $\nabla f(x, y) = \begin{bmatrix} 3x^2 + 3y \\ 3y^2 + 3x \end{bmatrix}$, the only critical points are where $x^2 = -y$ and $y^2 = -x$. It is easy to see that this implies $(x, y) = (0, 0)$ or $(x, y) = (-1, -1)$.

Now let’s consider the boundary pieces. Since $\frac{df}{dy} f(-2, y) = 3y^2 - 6$, we must include the point $(-2, -\sqrt{2})$. (Note that $y = \sqrt{2}$ is not within our region.) By symmetry, from the line $y = -2$ we must include the point $(-\sqrt{2}, -2)$. Next, since $\frac{df}{dy} f(1, y) = 3y^2 + 3$, we do not need to add any points. By symmetry, the same is true along the line $y = 1$. Finally, we must include all the corners, $(1, 1), (1, -2), (-2, 1), (-2, -2)$.

Here we check the values of $f$ at our selected points. $f(0, 0) = 0$, $f(-1, -1) = 1$, $f(-2, -\sqrt{2}) = f(-\sqrt{2}, -2) = -8 + 4\sqrt{2}$, $f(1, 1) = 4$, $f(1, -2) = f(-2, 1) = -13$, and $f(-2, -2) = -4$. Thus, the maximum value for $f$ is 4 which is attained at $(1, 1)$, and the minimum value for $f$ is $-13$ which is attained at $(1, -2)$ and $(-2, 1)$. 
(b) Match $f(x, y)$ with one of the following systems of level curves and EXPLAIN YOUR ANSWER.

The four figures are distinguished by whether $(0, 0)$ and $(-1, -1)$ are saddle points or extremal points. We can determine this for $f$ by using the second derivative test.

Observe that the Hessian matrix for $f$ at $(x, y)$ is

$$
\begin{bmatrix}
6x & 3 \\
3 & 6y
\end{bmatrix}
$$

Thus,

$$
\det(H_f(0,0)) = \det \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = -9
$$

and

$$
\det(H_f(-1,-1)) = \det \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} = 27.
$$

It follows that $f$ has a saddle point at $(0, 0)$ and an extremal point (i.e. a max or a min) at $(-1, -1)$. Therefore, the correct figure is figure I.
4. Let
\[ f(x, y) = \begin{cases} \frac{x^2y-x^3}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases} \]
Determine whether \( f \) is differentiable at \((0, 0)\).

Notice that \( f(x, 0) = -x \) and \( f(0, y) = 0 \). Therefore, \( Jf(0,0) = [-1 \ 0] \). Since if a function is differentiable its derivative is given by the Jacobian matrix, \( f \) is differentiable at \((0,0)\) if and only if the following limit is zero:

\[
\lim_{(a,b) \to (0, 0)} \frac{|f(a, b) - f(0, 0) - Jf(0,0) \begin{bmatrix} a \\ b \end{bmatrix}|}{\sqrt{a^2 + b^2}}
\]

This limit equals

\[
\lim_{(a,b) \to (0, 0)} \frac{|\frac{a^2b-a^3}{a^2+b^2} - a|}{\sqrt{a^2 + b^2}} = \lim_{(a,b) \to (0,0)} \frac{|a^2b + ab^2|}{(a^2 + b^2)^{3/2}}
\]

If we approach along the line \( a = b \) with \( a > 0 \), we get the limit

\[
\lim_{a \to 0^+} \frac{2a^3}{2^{3/2}a^3} = \sqrt{2} \neq 0.
\]
Since the limit along this line is zero, the overall limit cannot be zero. Therefore, \( f \) is NOT differentiable at \((0,0)\).
5. (a) Let \( g(x, y) = \ln(x^2 + y^2) \). Show that
\[
\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0
\]
We have \( g_x = \frac{2x}{x^2 + y^2} \), so \( g_{xx} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2} \). Similarly, \( g_y = \frac{2y}{x^2 + y^2} \), so \( g_{yy} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \). It follows that \( g_{xx} + g_{yy} = 0 \).

(b) Let \( f(x, y) = e^{x^2y-x-1} \). Find all directions in which the directional derivative of \( f \) is zero at the point \((1, 2)\). (Give unit vectors.)

\[
\nabla f(1, 2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.
\]
The directional derivative will be zero in all directions perpendicular to this vector. The unit vectors are
\[
\pm \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}.
\]

(c) Calculate the Jacobian matrix of \( h(x, y, z) = (xyz, y^2, x^2 + z^2) \) at \((1, 2, -1)\).

\[
Jh_{(x,y,z)} = \begin{bmatrix} yz & xz & xy \\ 0 & 2y & 0 \\ 2x & 0 & 2z \end{bmatrix} \quad \text{so} \quad Jh_{(1,2,-1)} = \begin{bmatrix} -2 & -1 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & -2 \end{bmatrix}
\]

(d) Find and classify the critical points of
\[
f(x, y, z) = \frac{1}{2}x^2 + \frac{3}{2}y^2 + z^2 + 2xy + 2xz + 3yz.
\]
Using \( \nabla f(x, y, z) = \begin{bmatrix} x + 2y + 2z \\ 3y + 2x + 3z \\ 2z + 2x + 3y \end{bmatrix} \), it is easy to see that the only critical point is at \((0, 0, 0)\). The Hessian matrix is
\[
H = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix}.
\]
Since \( \det(H) = 1 \), the eigenvalues are either \(+ + +\) or \(+ - -\). But \( \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = -1 \) so there is at least one negative eigenvalue. It follows that \( f \) has a saddle point at \((0,0,0)\).
6. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects across a certain plane, $\mathcal{P}$. Suppose that $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and we know that $T(\mathbf{w}) = \begin{bmatrix} 9 \\ 4 \\ 5 \end{bmatrix}$. Find an equation for $\mathcal{P}$.

The vector $T(\mathbf{w}) - \mathbf{w} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$ is perpendicular to the plane. Therefore, an equation for $\mathcal{P}$ is $4x + y + z = 0$. 
7. Let $M = AB$ where $A$ is a $4 \times 3$ matrix and $B$ is a $3 \times 3$ matrix.  
(a) Suppose that, for every $b$, $Mx = b$ has at most one solution. Prove that $B$ is invertible.

First notice that the null space of $M$ is trivial since otherwise $Mx = 0$ would have multiple solutions. Moreover, the null space of $M$ contains the null space of $B$. Therefore, the null space of $B$ must be trivial. Since any square matrix with trivial null space is invertible, it follows that $B$ is invertible.

(b) Suppose that the column space of $A$ is spanned by \[
\begin{bmatrix}
1 \\
-1 \\
2 \\
3 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
4 \\
-2 \\
\end{bmatrix},
\]
the null space of $A$ is spanned by \[
\begin{bmatrix}
2 \\
1 \\
1 \\
\end{bmatrix},
\]
the column space of $B$ is spanned by \[
\begin{bmatrix}
1 \\
0 \\
2 \\
\end{bmatrix}, \quad \begin{bmatrix}
-3 \\
3 \\
1 \\
\end{bmatrix}, \quad \begin{bmatrix}
5 \\
-2 \\
4 \\
\end{bmatrix}.
\]
Calculate the dimension of the column space of $M$.

It is easy to see that \[
\begin{bmatrix}
2 \\
1 \\
\end{bmatrix}
\]
is NOT in the span of \[
\begin{bmatrix}
1 \\
0 \\
2 \\
\end{bmatrix}
\]
and \[
\begin{bmatrix}
-3 \\
3 \\
1 \\
\end{bmatrix}.
\]
It follows that the null space of $A$ does not intersect the column space of $B$. Therefore, the null space of $M$ equals the null space of $B$ which is one dimensional. Thus, the column space of $M$ is 2 dimensional.
8. Suppose that we are opening a business manufacturing cans for the soup industry. These cans must be cylindrical and must hold a fixed volume \( V \). What ratio of height to radius would give a can with the least surface area (including the top and bottom) and thus require the least material.

Let the height and radius be \( h \) and \( r \), respectively. We wish to minimize the function
\[
A(h, r) = 2\pi rh + 2\pi r^2
\]
subject to the constraint
\[
V(h, r) = \pi r^2 h
\]
Using Lagrange multipliers, we must look at \( \nabla A = \lambda \nabla V \). That is,
\[
\begin{bmatrix}
2\pi r \\
2\pi h + 4\pi r
\end{bmatrix} = \lambda \begin{bmatrix}
\pi r^2 \\
2\pi rh
\end{bmatrix}
\]
Since clearly \( r \neq 0 \), the first equation implies \( \lambda = 2/r \). The second equation then becomes \( 2h + 4r = 4h \). Thus, \( h/r = 2 \). Since as we approach \( r \) or \( h \) equal to zero, the surface area blows up, this ratio must give the minimum. In other words, to make the can with the least material, the height should be twice the radius.
9. Let \( E(x, y) = x^3 + x - y^2 \). Consider the curve \( E(x, y) = 0 \). (This is called an elliptic curve.) Find all points \( p \) on the curve so that the tangent line through \( p \) goes through \((0, 0)\).

Let’s find the equation for the tangent line at the point \((a, b)\),

\[
\nabla E(a, b) = \begin{bmatrix} 3a^2 + 1 \\ -2b \end{bmatrix}.
\]

This is perpendicular to the tangent line. Thus, the equation is

\[
\begin{bmatrix} 3a^2 + 1 \\ -2b \end{bmatrix} \cdot \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) = 0
\]

Therefore, \((0, 0)\) is on the tangent line as long as

\[
3a^3 + a - 2b^2 = 0
\]

Of course, \((a, b)\) must also be in the curve so

\[
a^3 + a - b^2 = 0
\]

It is easy to see that the only points which satisfy these two conditions are \((0, 0), (1, \sqrt{2})\) and \((1, -\sqrt{2})\).
10. (a) Suppose that \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) form a basis for a certain vector space. Do \( \mathbf{u} - 2\mathbf{v}, \mathbf{v} + \mathbf{w}, \) and \( \mathbf{u} - \mathbf{v} + \mathbf{w} \) also form a basis? Explain.

No. In order to form a basis, this collection of vectors must be linearly independent. However,

\[
(\mathbf{u} - 2\mathbf{v}) + (\mathbf{v} + \mathbf{w}) - (\mathbf{u} - \mathbf{v} + \mathbf{w}) = 0.
\]

(b) Let \( \mathbf{v} \) and \( \mathbf{w} \) be vectors of equal length. Prove that \( \mathbf{v} + \mathbf{w} \) is perpendicular to \( \mathbf{v} - \mathbf{w} \).

It is enough to show that the dot product of these vectors is zero. We have

\[
(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}
= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w}
= ||\mathbf{v}||^2 - ||\mathbf{w}||^2
= 0
\]