MATH 51
SOLUTIONS TO FIRST SAMPLE MIDTERM #2

90 Minutes

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. No calculators.

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1. Compute the Jacobian matrix for each of the following functions, \( f \), at the indicated point \( p \).
   (a) \( f(x,y,z) = (x^2y, zy), \ p = (1,2,3) \)
   
   \[
   Jf_{(x,y,z)} = \begin{bmatrix} 
   2xy & x^2 & 0 \\
   0 & z & y \\
   \end{bmatrix}.
   \]
   So, \( Jf_p = \begin{bmatrix} 
   4 & 1 & 0 \\
   0 & 3 & 2 \\
   \end{bmatrix} \).

   (b) \( f(x,y) = xy \ln(xy^2), \ p = (e, -1) \)
   
   \[
   Jf_{(x,y)} = \begin{bmatrix} 
   y + y \ln(xy^2) & 2x + x \ln(xy^2) \\
   -2 & 3e \\
   \end{bmatrix}.
   \]
   So, \( Jf_{(x,y)} = \begin{bmatrix} 
   y + y \ln(xy^2) & 2x + x \ln(xy^2) \\
   -2 & 3e \\
   \end{bmatrix} \).

   (c) \( f(x,y) = (\cos(\pi xy), x \sin(\pi y)), \ p = (1/3, 1/2) \)
   
   \[
   Jf_{(x,y)} = \begin{bmatrix} 
   -\pi y \sin(\pi xy) & -\pi x \sin(\pi xy) \\
   \sin(\pi y) & \pi x \cos(\pi y) \\
   \end{bmatrix}.
   \]
   So, \( Jf_{(x,y)} = \begin{bmatrix} 
   -\pi y \sin(\pi xy) & -\pi x \sin(\pi xy) \\
   \sin(\pi y) & \pi x \cos(\pi y) \\
   \end{bmatrix} \).
2. A woodlouse is crawling around on a table which is illuminated by a lamp. Woodlice do not like light, and so move away from it as rapidly as possible. Assume that the illumination at the point \((x, y)\) is given by the function \(I(x, y) = 1000 - (x^2 + 4y^2)\) and that the woodlouse starts at the point \((5, 3)\). (All distances are in centimeters.)

(a) In the direction of what vector does the woodlouse move initially?

The woodlouse (let’s call him Steve) moves in the direction of \(-\nabla f(x, y)\). We have

\[
\nabla f(x, y) = \begin{bmatrix} -2x \\ -8y \end{bmatrix}, \quad \nabla f(5, 3) = \begin{bmatrix} -10 \\ -24 \end{bmatrix}.
\]

So, Steve moves in the direction of \(\begin{bmatrix} 10 \\ 24 \end{bmatrix}\).

(b) From the point of view of the woodlouse, at what rate (per unit time) does the illumination drop initially. Note that the woodlouse crawls at a speed of 0.1 cm/sec.

In the direction of \(\nabla f(5, 3)\), the illumination increases at \(||\nabla f(5, 3)|| = \sqrt{10^2 + 24^2} = 26\) units/cm. Therefore, in the direction of \(-\nabla f(5, 3)\) it decreases at this rate.

We want to know how fast the illumination drops per unit time. Since Steve crawls at 0.1 cm/sec, the illumination declines at 2.6 units/sec.
3. Suppose that \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) is a differentiable function which satisfies \( f(1, 2) = \left( \frac{3}{4}, \frac{1}{2} \right) \) and \( Jf_{(1, 2)} = \begin{bmatrix} -1 & -7 \\ 3 & -4 \end{bmatrix} \). Given this information, find the best estimate for a point \((x, y)\) that satisfies \( f(x, y) = (0, 0)\). Explain your reasoning.

Suppose \( \mathbf{v} \) satisfied \( Jf_{(1, 2)} \mathbf{v} = \begin{bmatrix} -1/4 \\ -1/2 \end{bmatrix} \). Then

\[
 f((1, 2) + \mathbf{v}) \approx f(1, 2) + Jf_{(1, 2)} \mathbf{v} \\
= \left( \frac{1}{4}, \frac{1}{2} \right) + \begin{bmatrix} -1/4 \\ -1/2 \end{bmatrix} \\
= (0, 0).
\]

Thus, \((1, 2) + \mathbf{v}\) would be the best linear estimate for the point we are looking for.

To find \( \mathbf{v} \), we do elimination on the augmented matrix:

\[
\begin{bmatrix}
-1 & -7 & -1/4 \\
3 & -4 & -1/2 \\
1 & 7 & 1/4 \\
0 & -25 & -5/4 \\
1 & 0 & -1/10 \\
0 & 1 & 1/20
\end{bmatrix}
\]

Thus, \( \mathbf{v} = \begin{bmatrix} -1/10 \\ 1/20 \end{bmatrix} \) and the point we want is \( \left( \frac{3}{7}, \frac{11}{20} \right) + \begin{bmatrix} -1/10 \\ 1/20 \end{bmatrix} = \left( \frac{3}{20}, \frac{11}{20} \right) \).
4. (a) Let \( f(x, y) = e^{x^2 y} + \sin(xy) + x^2 \). Find an equation for the tangent plane to the graph of \( f(x, y) \) when \( (x, y) = (2, 0) \).

Note that \( f(2, 0) = 5 \). Let \( g(x, y, z) = e^{x^2 y} + \sin(xy) + x^2 - z \). We want the tangent plane to the level surface \( g(x, y, z) = 0 \) at \( (2, 0, 5) \). We have

\[
\nabla g(x, y, z) = \begin{bmatrix} 2x ye^{x^2 y} + y \cos(xy) + 2x \\ x^2 e^{x^2 y} + x \cos(xy) \\ -1 \end{bmatrix}.
\]

Thus,

\[
\nabla g(2, 0, 5) = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}.
\]

The plane perpendicular to \( \nabla g(2, 0, 5) \) and through the point \( (2, 0, 5) \) is

\[
4x + 6y - z = 8 + 0 - 5 = 3.
\]

That is,

\[
4x + 6y - z = 3.
\]

(b) Suppose that \( f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a differentiable function so that

\[
f_x(5, 6) = 3, \quad f_y(5, 6) = -2, \quad f_{xx}(5, 6) = 0,
\]

\[
f_{xy}(5, 6) = f_{yx}(5, 6) = 1, \quad f_{yy}(5, 6) = 2.
\]

Let \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined by

\[
g(u, v) = (u + v, uv).
\]

Find \( \frac{\partial^2}{\partial v \partial u} (f \circ g) \) at \((3, 2)\).

Note that, for any function \( h(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R} \),

\[
\nabla h \circ g(u, v) = \begin{bmatrix} h_x(g(u, v)) \\ h_y(g(u, v)) \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix}.
\]

In particular,

\[
\frac{\partial}{\partial u} f \circ g = f_x(g) + vf_y(g).
\]

Thus,

\[
\frac{\partial^2}{\partial v \partial u} f \circ g = \frac{\partial}{\partial v} f_x(g) + f_y(g) + v \frac{\partial}{\partial v} f_y(g).
\]

Using our initial formula with \( h = f_x \) and \( h = f_y \), we get

\[
\frac{\partial^2}{\partial v \partial u} f \circ g = f_{xx}(g) + uf_{xy}(g) + f_y(g) + vf_{yx}(g) + uvf_{yy}(g).
\]

Since \( g(3, 2) = (5, 6) \), evaluating gives

\[
f_{xx}(5, 6) + 3f_{xy}(5, 6) + f_y(5, 6) + 2f_{yx}(5, 6) + 6f_{yy}(5, 6) = 0 + 3 - 2 + 2 + 12 = 15
\]
5. In the math 51 ice cream parlor, ice cream is served in cylindrical cups. It is served so as to entirely fill the cup and there is a scoop on top in the shape of a half-sphere of radius half the radius of the cup. Suppose that I have only a limited amount of paper to make my cup, but I want to get as much ice cream as possible. What kind of cup should I make? In other words, what ratio of height to radius should I use?

The amount of paper needed is

\[ A(r, h) = \pi r^2 + 2\pi rh. \]

This is constant. The amount of ice cream is

\[ V(r, h) = \pi r^2 h + \frac{2}{3} \pi (r/2)^3. \]

This is what we want to maximize. By the method of Lagrange multipliers, in the optimal design, \( r \) and \( h \) are so that the gradients of \( A \) and \( V \) are parallel. That is,

\[
\begin{bmatrix}
2\pi rh + \frac{2}{3} r^2 \\
\pi r^2 \\
\end{bmatrix} = \lambda \begin{bmatrix}
2\pi r + 2\pi h \\
2\pi r \\
\end{bmatrix}.
\]

Since \( r \) obviously can’t be zero, we can cancel \( r \) in the equation \( \pi r^2 = 2\pi \lambda r \) and get \( \lambda = r/2 \). Therefore,

\[ 2\pi rh + \frac{\pi}{4} r^2 = \pi r^2 + \pi rh. \]

That is,

\[ \pi rh = \frac{3}{4} \pi r^2. \]

Canceling \( r \pi \) gives

\[ h = \frac{3}{4} r. \]

Thus, the optimal design has ratio of height to radius equal \( 3/4 \).
6. Let \( f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases} \)

(a) Prove that \( f \) is continuous at \((0, 0)\).

\[
\lim_{(x,y) \to (0,0)} \left| \frac{x^3}{x^2 + y^2} \right| = \lim_{(x,y) \to (0,0)} \left| x \right| \left| \frac{x^2}{x^2 + y^2} \right| 
\leq \lim_{(x,y) \to (0,0)} \left| x \right| = 0.
\]

Since \( f(0,0) = 0 \), this shows that \( f \) is continuous at \((0,0)\).

(b) Compute the partial derivatives of \( f \) at \((0,0)\).

\[
f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} = 1
\]
\[
f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0
\]

(c) Show that \( f \) is not differentiable at \((0,0)\).

By (b), \( Jf(0,0) = [1 \ 0] \). Let \( \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \). Then,

\[
\lim_{\mathbf{v} \to 0} \frac{f((0,0) + \mathbf{v}) - f(0,0) - Jf(0,0)\mathbf{v}}{||\mathbf{v}||} = \lim_{(a,b) \to (0,0)} \frac{\frac{a^3}{a^2 + b^2} - a}{\sqrt{a^2 + b^2}}
\]
\[
= \lim_{(a,b) \to (0,0)} \frac{-ab^2}{(a^2 + b^2)^{3/2}}
\]

Along \( a = 0 \) or \( b = 0 \) this limit is zero. However, along \( a = b \), we get

\[
\lim_{a \to 0} \frac{-a^3}{(2a^2)^{3/2}} = -2^{-3/2} \neq 0.
\]

Thus, our original limit does not exist. Therefore, \( f \) is not differentiable at \((0,0)\).