MATH 51
FIRST SAMPLE MIDTERM #1
SOLUTIONS

90 Minutes

NAME:

Section Number:

I agree to abide by the Honor Code.
Signature:

Instructions: Show all work. No calculators.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
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<tbody>
<tr>
<td>1.</td>
<td>18</td>
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<td>Total</td>
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1. For each of the following, either give an example of a matrix, $M$, which satisfies the conditions, or else explain why it can’t exist.

(a) \[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}
\text{ and } \begin{bmatrix}
2 \\
3 \\
2 \\
1
\end{bmatrix}
\] span the column space of $M$, and \[
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\] spans the null space of $M$.

$M$ must be $4 \times 3$. For two of its columns we can take the given basis of its null space. In order to get \[
\begin{bmatrix}
1 \\
4 \\
-14
\end{bmatrix}
\] in the null space, we need to take the third column equal to $-2$ times the first $-3$ times the second. Thus, \[
M = \begin{bmatrix}
1 & 4 & -14 \\
2 & 3 & -13 \\
3 & 2 & -12 \\
4 & 1 & -11
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}
\text{ and } \begin{bmatrix}
6 \\
5 \\
4 \\
2
\end{bmatrix}
\] span the column space of $M$, and \[
\begin{bmatrix}
2 \\
3 \\
4 \\
5
\end{bmatrix}
\] spans the null space of $M$.

This is not possible. $M$ would be a $6 \times 4$ matrix with a two-dimensional column space and a one-dimensional null space. But the dimensions of the null and column space must add up to the number of columns (4 in this case).

(c) $M$ is $3 \times 2$ and \[
\begin{bmatrix}
1 \\
2 \\
2
\end{bmatrix}, \begin{bmatrix}
-1 \\
1 \\
-1
\end{bmatrix}, \text{ and } \begin{bmatrix}
-2 \\
11 \\
1
\end{bmatrix}
\] span the column space.

The matrix \[
\begin{bmatrix}
1 & -1 & -2 \\
2 & 1 & 11 \\
2 & -1 & 1
\end{bmatrix}
\] has the correct null space, but not the right shape. Doing row-reduction gives \[
\begin{bmatrix}
1 & -1 & -2 \\
0 & 1 & 5 \\
0 & 0 & 0
\end{bmatrix}
\].

Thus, the column space is spanned by the first two vectors alone. It follows that $M = \begin{bmatrix}
1 & -1 \\
2 & 1 \\
2 & -1
\end{bmatrix}$ has the correct null space.
2. Consider the following system of linear equations

\[
\begin{align*}
2x + y &= 2 \\
-2x - 2y + z &= -4 \\
3x + z &= 2
\end{align*}
\]

(a) Re-write the system as a matrix equation.

\[
\begin{bmatrix}
2 & 1 & 0 \\
-2 & -2 & 1 \\
3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
2 \\
-4 \\
2
\end{bmatrix}
\]

(b) Find the inverse of the matrix from (a).

We must compute the reduced echelon form of

\[
\begin{bmatrix}
2 & 1 & 0 & 1 & 0 & 0 \\
-2 & -2 & 1 & 0 & 1 & 0 \\
3 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Eliminating in the first column gives

\[
\begin{bmatrix}
1 & 1/2 & 0 & 1/2 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 \\
0 & -3/2 & 1 & -3/2 & 0 & 1
\end{bmatrix}
\]

Next, we eliminate in the second column:

\[
\begin{bmatrix}
1 & 0 & 1/2 & 1 & 1/2 & 0 \\
0 & 1 & -1 & -1 & -1 & 0 \\
0 & 0 & -1/2 & -3 & -3/2 & 1
\end{bmatrix}
\]

Finally, eliminate in the third column:

\[
\begin{bmatrix}
1 & 0 & 0 & -2 & -1 & 1 \\
0 & 1 & 0 & 5 & 2 & -2 \\
0 & 0 & 0 & 6 & 3 & -2
\end{bmatrix}
\]

Thus,

\[
M^{-1} = \begin{bmatrix}
-2 & -1 & 1 \\
5 & 2 & -2 \\
6 & 3 & -2
\end{bmatrix}
\]

(c) Using the answer to (b), solve the system.

We have

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= M^{-1}
\begin{bmatrix}
2 \\
-4 \\
2
\end{bmatrix}
= \begin{bmatrix}
-2 & -1 & 1 \\
5 & 2 & -2 \\
6 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
-4 \\
2
\end{bmatrix}
= \begin{bmatrix}
2 \\
-2 \\
-4
\end{bmatrix}
\]
3. Find a $3 \times 3$ matrix that rotates the $x$-$y$ plane by an angle $\theta$ and which sends the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$.

Just as when rotating in $\mathbb{R}^2$, it is easy to see in this case that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ gets mapped to $\begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}$. Similarly, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. It follows that the matrix is

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$
4. Give equations for two planes which intersect in the line
\[
\begin{bmatrix}
3 \\
-2 \\
2
\end{bmatrix} + t \begin{bmatrix}
1 \\
2 \\
4
\end{bmatrix}.
\]

First, let’s get planes which intersect in the span of \( \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \).

For this, we must find two independent vectors perpendicular to this one. It is easy to see that \( \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \) work.

Therefore, the planes
\[
\begin{bmatrix}
-4 \\
0 \\
-2
\end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}
-2 \\
2 \\
0
\end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0
\]

intersect in the line through the origin.

Next, we shift everything by \( \begin{bmatrix} -2 \\ 2 \end{bmatrix} \). We get the planes
\[
\begin{bmatrix}
-4 \\
0 \\
1
\end{bmatrix} \cdot \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right) = 0 \quad \text{and} \quad \begin{bmatrix}
-2 \\
2 \\
0
\end{bmatrix} \cdot \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right) = 0.
\]

In other words, the planes
\[
\begin{align*}
-4x + z &= -10 \\
-2x + y &= -8
\end{align*}
\]

intersect in the given line.
5. Suppose that $A$ is a $3 \times 3$ matrix, $B$ is a $3 \times 4$ matrix, and $M = AB$.
(a) Suppose that the column space of $M$ is three-dimensional. Prove that $A$ is invertible.

We know that the column space of $A$ contains the column space of $M$. Since the column space of $M$ is three-dimensional, the column space of $A$ is at least three-dimensional. But $A$ is $3 \times 3$. Therefore, the column space of $A$ is as large as it could possibly be, namely all of $\mathbb{R}^3$. But any square matrix with linearly independent columns is invertible.

(b) Suppose that the null space and the column space of $A$ are spanned by \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \), respectively. The null space and the column space of $B$ are spanned by \( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \), and \( \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \end{bmatrix} \), respectively. Compute the dimension of the null space of $M$.

The null space of $M$ contains the null space of $B$ which is two-dimensional. Thus, the null space of $M$ is at least two-dimensional. But it could be larger. It is larger if the column space of $B$ has non-trivial intersection with the null space of $A$. Since the null space of $A$ is one-dimensional, we see that the dimension of the null space of $M$ is either two or three.

To decide which it is, we must determine whether \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is in the span of \( \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 7 \\ 1 \end{bmatrix} \). We do Gaussian elimination on the column space of $B$ and so the null space of $M$ is three-dimensional.
6. Suppose that \( M \) is an \( n \times n \) matrix with the following property: for any \( \mathbf{v}, \mathbf{w} \),
\[
(M\mathbf{v}) \cdot (M\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}.
\]
(Such a matrix is called \textit{orthogonal}.)
(a) Prove that the columns of \( M \) form an orthonormal basis for \( \mathbb{R}^n \).

Let \( \mathbf{e}_1, \ldots, \mathbf{e}_n \) be the standard basis for \( \mathbb{R}^n \). The \( i \)th column of \( M \) is \( M\mathbf{e}_i \). Furthermore, since \( M \) is an orthogonal matrix, for every \( i \)
\[
(M\mathbf{e}_i) \cdot (M\mathbf{e}_i) = \mathbf{e}_i \cdot \mathbf{e}_i = 1
\]
and, for \( i \neq j \),
\[
(M\mathbf{e}_i) \cdot (M\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = 0.
\]
But this is exactly what it means for the columns of \( M \) to be orthonormal.

(b) Prove that \( M \) is invertible.

To prove that \( M \) is invertible, it is enough to check that the columns of \( M \) are linearly independent. But ANY collection of orthonormal vectors is linearly independent. To see this, suppose that \( \mathbf{u}_1, \ldots, \mathbf{u}_n \) are orthonormal. If
\[
a_1 \mathbf{u}_1 + \cdots + a_n \mathbf{u}_n = \mathbf{0},
\]
then, for each \( i \), we take the dot product of both sides with \( \mathbf{u}_i \). We get \( a_i = 0 \). This shows that the \( \mathbf{u}'s \) are linearly independent.