More Examples of Planes

There are three ways to describe a plane in $\mathbb{R}^3$: with a single linear equation, parametrically (i.e. with two spanning vectors and a translating vector), or using a normal vector and a translating vector. Here we give some examples of converting between these representations.

The set of vectors perpendicular to a given vector, $\mathbf{n}$, is the set of all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $\mathbf{n} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. This is a plane through the origin. Now suppose we translate by $\mathbf{q}$. How do we find an equation for the resulting plane?

Here is an analogy. Suppose we have a device that beeps when it is pointing straight north. How can we recognize when we are pointing straight west? Well, to go from pointing north to pointing west we rotate $90^\circ$ to the left. Therefore, we are pointing west if, when we turn $90^\circ$ to the right, the device beeps.

In the same way, how can we recognize that we are on the plane translated by $\mathbf{q}$? If, when we translate back by $\mathbf{q}$, we are on the original plane. Therefore, $\mathbf{r}$ is on the translated plane if $\mathbf{r} - \mathbf{q}$ is perpendicular to $\mathbf{n}$, i.e. if $\mathbf{n} \cdot (\mathbf{r} - \mathbf{q}) = 0$. This can also be written as $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{q}$.

EXAMPLE. Let $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. An equation for the plane through $\mathbf{q}$ perpendicular to $\mathbf{n}$ is

$$\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \cdot \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = 0$$

i.e.

$$\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus, $3x - y + z = 2$.

EXAMPLE. Find an equation for the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

All we need is a normal vector, that is, any non-zero vector $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ which satisfies $\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$. This leads to a
system of two equations in three variables
\[ n_1 + n_2 + n_3 = 0 \]
\[ n_1 + 2n_2 + 3n_3 = 0. \]
Subtracting,
\[ n_1 + n_2 + n_3 = 0 \]
\[ n_2 + 2n_3 = 0. \]
There are, of course, a whole lines worth of normal vectors, but we only want one non-zero one. Set \( n_3 = 1 \). Then we get \( n_2 = -2 \) and \( n_1 = 1 \).
Therefore \( \mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \) is a normal vector for our plane. An equation for the plane is \( x - 2y + z = 0 \).

EXAMPLE. Consider the plane \( x - 3y + 4z = 0 \). This plane can also be described as the set of vectors perpendicular to \( \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \). Let’s find two vectors which span the plane. For this, it is enough to find two linearly independent vectors on the plane. Notice that \( x = -4, y = 0, z = 1 \) and \( x = 3, y = 1, z = 0 \) are solutions to the equation. Thus, the plane is spanned by \( \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \).

Next, consider the plane \( x - 3y + 4z = 3 \). This plane is parallel to the previous one. To give a parametric description, we must find one vector on the plane, e.g. \( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \). The plane consists of all \( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \), \( s, t \) any scalars.

EXAMPLE. Find a system of two linear equations whose set of solutions is the line spanned by \( \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \). In other words, find two planes whose intersection is this line. Since \( \mathbf{w} \) is on both planes, it will be perpendicular to both normal vectors. In other words, we need two independent vectors perpendicular to \( \mathbf{w} \). In this case, we can find them by inspection: \( \mathbf{n}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \) and \( \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \) work. Thus, our line is the set of solutions to
\[ x - 2y = 0 \]
\[ y - z = 0. \]