Problem File 2 – limited picture version

1. Let $f(x, y) = (\sqrt{xy}, x\ln y)$. The derivative of $f$ at the point $p = (4, 1)$ is the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$J = \begin{bmatrix} 1/4 & 1 \\ 0 & 4 \end{bmatrix}.$$  

For each of the following displacement vectors $v$, calculate (1) $f(p + v)$, and (2) $f(p) + Jv$.

(a) $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  
(b) $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  
(c) $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Describe (and, if possible, sketch) the domains of each of the following functions

(a) $f(x, y) = (\sqrt{x+\frac{1}{y}}, \frac{1}{x+y})$
(b) $g(x, y, z) = \sqrt{1-x^2-y^2-z^2}$
(c) $h(x, y) = (xy, \ln(1+x), e^{-y})$

3. The four pictures below are each images of the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ under a certain function. Match each of the following functions with the image of the unit square under that function.

(a) $f(x, y) = (x - y, x + y)$
(b) $f(x, y) = (x, y)$
(c) $f(x, y) = (x, xy)$
(d) $f(x, y) = (x^2 - y^2, y)$
4. The four pictures in picture file 1 are each images of the unit sphere under a certain function. Match each of the following functions with the image of the unit sphere under that function.

(a) \( f(x, y, z) = (x^3, y, z) \)
(b) \( f(x, y, z) = (x, y, z + x^2 + y^2) \)
(c) \( f(x, y, z) = (x, y, z) \)
(d) \( f(x, y, z) = (x + z^2, y, z) \)

5. In picture files 2 and 3 there are graphs of 8 functions above the region \(-1 \leq x \leq 1, -1 \leq y \leq 1\). Match each of the following functions to its graph.

(a) \( f(x, y) = \cos(\pi x) \cos(\pi y) \)
(b) \( f(x, y) = y \sin(2\pi x) \)
(c) \( f(x, y) = x^2 y \)
(d) \( f(x, y) = xy^4 \)
(e) \( f(x, y) = 1 - x^2 - y^2 \)
(f) \( f(x, y) = x^2 y^3 \)
(g) \( f(x, y) = e^{-8(x^2+y^2)} \)
(h) \( f(x, y) = xy^3 \)

6. In picture file 4, there are 4 collections of level curves and 4 graphs.

Match each set of level curves with the corresponding graph.
7. Match each of the following functions with the picture of its vector field on the region $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

(a) $f(x, y) = (xy, 1)$
(b) $f(x, y) = (y, x)$
(c) $f(x, y) = (x, \sin(2\pi y))$
(d) $f(x, y) = (-x, y^2)$
(e) $f(x, y) = (x, y)$
(f) $f(x, y) = (x + y, x - y)$

Vector Field I

Vector Field II

Vector Field III

Vector Field IV

Vector Field V

Vector Field VI
8. Let
\[ f(x, y) = \begin{cases} \frac{x^2 + y^2}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases} \]
(The graph of \( f \) is given in picture file 5.)
(a) Prove that \( f \) is continuous at \((0, 0)\).
(b) Compute \( \frac{\partial f}{\partial x}(0, 0) \) and \( \frac{\partial f}{\partial y}(0, 0) \).
(c) Show that \( f \) is NOT differentiable at \((0, 0)\).

9. Let \( A \) and \( B \) be \( 2 \times 2 \) matrices. Prove:
(a) \( \det AB = (\det A)(\det B) \)
(b) \( \det A = 1/\det A^{-1} \)
(c) if \( A = MBM^{-1} \) then \( \det A = \det B \)

10. Each of the following functions has a critical point at \((0, 0)\). Compute the Hessian matrix at that point, and then express it in terms of the given basis. Use the result to classify the critical point.
(a) \( \frac{3}{2}x^2 + 2xy; \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \)
(b) \( \frac{3}{2}x^2 + 2xy + 3y^2; \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \)

11. Find, and use the second-derivative test to classify, the critical points of the following functions.
(a) \( f(x, y) = x^2 + 4xy + 2y^2 + 4x - 8y - 1 \)
(b) \( f(x, y) = x^3 + y^3 + 3xy + 2 \)
(c) \( f(x, y) = 8xy - 2x^2 - y^4 \)
(d) \( f(x, y) = e^{x^2y^2-12x+y} \)

12. Compute the determinants of the following matrices.
(a) \( \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \)
(b) \( \begin{bmatrix} 3 & 1 & 0 \\ 4 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix} \)

13. Each of the following functions has a critical point at \((0, 0, 0)\). Use the second derivative test to decide if it is a local maximum, local minimum, or a saddle point. If it has a saddle point, indicate whether it has two positive eigenvalues or two negative eigenvalues.
(a) \( f(x, y, z) = \frac{1}{2}(x + y)^2 + \frac{1}{2}(x - z)^2 + z^2 \)
(b) \( f(x, y, z) = -x^2 - y^2 - 2z^2 + xy + xz - 2yz \)
(c) \( f(x, y, z) = 2x^2 + y^2 + 2z^2 - 3xy + yz \)