**Geometry of vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)**

Here we begin to study geometric interpretations of vectors. Keep in mind that, although geometric interpretation is easiest in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) (so we can draw pictures), linear algebra works the same for all vector spaces.

There are two standard interpretations of a 2-vector such as \(
\begin{bmatrix}
2 \\
4
\end{bmatrix}
\). One is as the point \((2, 4)\) on a two-dimensional coordinate system. The other is as an arrow going from \((0, 0)\) to \((2, 4)\).

Vector addition is fairly easy to picture: \( \mathbf{v} + \mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \).
Next, let’s consider subtraction: \( \mathbf{u} - \mathbf{w} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \)

Notice that \( \mathbf{u} - \mathbf{w} \) is parallel to the line between \( \mathbf{u} \) and \( \mathbf{w} \).

What about scalar multiplication? \( 2\mathbf{w} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, 3\mathbf{w} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, (-1)\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)

It is now clear that the set of all scalar multiples of \( \mathbf{w} \) (i.e. the span of \( \mathbf{w} \)) is the line through \((0,0)\) in the direction of \( \mathbf{w} \):
Note that here we are using the interpretation of vectors as points – the collection of all scalar multiples of \( \mathbf{w} \) is exactly the collection of points on the line.

Next, we combine vector addition and scalar multiplication. Let’s consider \( \mathbf{u} + t \mathbf{w} \) for \( t \) any scalar. We get a line parallel to the span of \( \mathbf{w} \), but translated so it now goes through \( \mathbf{u} \).

The expression \( \mathbf{u} + t \mathbf{v} \) is called a parametric representation of the line. Note that the parametric representation of the line is not unique. For example, in place of \( \mathbf{u} \) we could take any other vector on the line.

Below we give an example involving the span of \( \mathbf{u} \) together with the span of \( \mathbf{u} \) translated by \( \mathbf{w} \), that is, the line \( \mathbf{w} + t \mathbf{u}, t \) any scalar:
The observation that a line not through the origin arises as a translation of a line through the origin, has an exact parallel in the theory of linear equations. For example, let's find all solutions to the following system:

\[
\begin{align*}
  x + 3y &= 0 \\
  -x - 3y &= 0 \\
  -3x - 9y &= 0
\end{align*}
\]

The corresponding augmented matrix is

\[
\begin{bmatrix}
  1 & 3 & 0 \\
  -1 & -3 & 0 \\
  -3 & -9 & 0
\end{bmatrix}
\]

After Gaussian elimination we have

\[
\begin{bmatrix}
  1 & 3 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]

Thus, there are infinitely many solutions; \( y \) can be any number and then \( x = -3y \). If we form \( x, y \) into a column vector, we have \( \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \) for any number \( t \). In other words, the solutions of this system of equations can be viewed as the line span of \( \mathbf{u} \). (\( \mathbf{u} \) as above.) Now consider the same system of equations except with non-zero constants on right. (This is called a non-homogeneous system. When all constants are zero, a system is homogeneous.)

\[
\begin{align*}
  x + 3y &= -2 \\
  -x - 3y &= 2 \\
  -3x - 9y &= 6
\end{align*}
\]
The corresponding augmented matrix is

\[
\begin{bmatrix}
1 & 3 & -2 \\
-1 & -3 & 2 \\
-3 & -9 & 6
\end{bmatrix}
\]

After Gaussian elimination we have

\[
\begin{bmatrix}
1 & 3 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Again \( y \) can be any number and then \( x = -2 - 3y \). Thus, the solutions are \( \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \) for any number \( t \). As before, we get a line of solutions. However, this time it is not through the origin – the set of all solutions is span of \( \mathbf{u} \) translated by one fixed solution. In fact, it always happens that the set of solutions of a non-homogeneous system of linear equations is the translation of the solutions of the corresponding homogeneous system by any one solution. Since it is easy to see that \( \mathbf{w} \) is a solution to the non-homogeneous system, another parametric expression for the line of solutions is \( \mathbf{w} + t\mathbf{u} \).

***

The geometry of vectors in \( \mathbb{R}^3 \) works similarly. The span of a single non-zero vector is a line. Let \( \mathbf{u}, \mathbf{v} \) be two vectors in \( \mathbb{R}^3 \). If they are multiples of each other, then together they still span a line (and so are linearly dependent). If they are not multiples of each other, they span a plane:
Including a third vector \( \mathbf{w} \) may not increase the span:

Or, it may make the span all of \( \mathbb{R}^3 \):

In this case, we can translate the span of \( \mathbf{u}, \mathbf{v} \) by \( \mathbf{w} \) and get a plane not through the origin:
$w + c_1 u + c_2 v$

c_1 u + c_2 v