Homework 5 Solutions
Course Reader Exercises

104. (a)

\[(C^{-1}AC)^m = (C^{-1}AC)(C^{-1}AC)(C^{-1}AC)\ldots(C^{-1}AC)\]
\[= C^{-1}A(CC^{-1})A(CC^{-1})A\ldots(CC^{-1})AC\]
\[= C^{-1}AI_nA \cdots I_nAC\]
\[= C^{-1}A^mC\]

since \(CC^{-1} = I_n\) and \(I_nA = A\).

(b) If \(B = C^{-1}AC\), then

\[CBC^{-1} = CC^{-1}ACC^{-1} = I_nAI_n = A\]

105. (a) \(C^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\)

(b) \(C^{-1}AC = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\)

(c) \(C^{-1}A^7C = (C^{-1}AC)^7 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^7 = \begin{bmatrix} 3^7 & 0 \\ 0 & 2^7 \end{bmatrix}\)

so

\[A^7 = C \begin{bmatrix} 3^7 & 0 \\ 0 & 2^7 \end{bmatrix} C^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3^7 & 0 \\ 0 & 2^7 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\]

\[= \begin{bmatrix} 1 \cdot 3^7 - 2 \cdot 2^7 \\ -1 \cdot 3^7 + 2 \cdot 2^7 \end{bmatrix} \begin{bmatrix} 2^8 - 2(3^7) \\ 3^7 - 2^7 \end{bmatrix} = \begin{bmatrix} -1931 & -4118 \\ 2059 & 4246 \end{bmatrix}\]

106 There are many reasons why this is true.

**reason 1** Since the determinant of \(A\) is the (signed) area of the parallelogram formed by the vectors \[\begin{bmatrix} a \\ c \end{bmatrix}\] and \[\begin{bmatrix} b \\ d \end{bmatrix}\], the only way the area could be zero is if the two vectors span a line or a point. i.e. – they must be linearly dependent.

But if the columns of \(A\) are linearly dependent, then \(A\) is not a one-to-one map. Geometrically, it maps the plane to a line or point, so it must send something to zero. Or, you could say that if the columns are linearly dependant, then when you row reduce, you must get a free variable. But each free variable adds one dimension to the nullspace. If the nullspace has dimension \(\geq 1\), that means that
there is a vector which is sent to the zero vector. That means that \( A \) can’t be one-to-one, since we already know it sends the zero vector to itself. Therefore \( A \) is at least two-to-one. Actually, and you should understand this, \( A \) will send an infinite number of vectors to zero. Namely the whole nullspace is sent to the zero vector.

**reason 2** Fudge around with algebra. We already know that the rank of the matrix must be less than 2, so there ought to be a way of showing that. First, assume that, say, \( a \neq 0 \) and \( c \neq 0 \). Now, multiply the first row by \( c \) and the second by \( a \), then our new matrix is

\[
\begin{bmatrix}
ca & cb \\
ac & ad
\end{bmatrix}
\]

But \( ad = bc \), so letting \( R_2 = R_2 - R_1 \)

\[
\begin{bmatrix}
0 & ad - bc \\
ac & ad
\end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}
\]

Thus we get a row of zeros, which means we get a free variable, which means the matrix is not one-to-one as above.

Of course you need to take care of the case when \( a = 0 \) or when \( c = 0 \). Also note that reason 2 is much more “low-level” than reason 1. We don’t use any knowledge of spans or that determinants compute volumes. All that geometry is pushed behind the scenes. The price we pay is a level of computational trickery. This is true of life in general. Things that seems like computational magic usually have simple geometric explanations. To wit,

**reason 3** just compute:

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
d \\
-c
\end{bmatrix} = \begin{bmatrix}
ad - bc \\
0
d - dc
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

So we have shown that the vector \( A \) sends \( \begin{bmatrix} d \\
-c
\end{bmatrix} \) to the zero vector. Now if

\[
\begin{bmatrix} d \\
-c
\end{bmatrix} \neq \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

then \( A \) is not one-to-one. On the other hand, if

\[
\begin{bmatrix} d \\
-c
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

then \( d = c = 0 \). In that case, we also notice that

\[
\begin{bmatrix}
a & b \\
0 & 0
\end{bmatrix} \begin{bmatrix}
-b \\
a
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

Thus

\[
\begin{bmatrix}
-b \\
a
\end{bmatrix} \in N(A)
\]
So either both \( b \) and \( a \) are zero or \( A \) is not one-to-one. Thus \( A \) is not one-to-one.

**reason 4** Now the last reason requires the least knowledge of linear algebra, and therefore is the most cryptic and tricky of them all.

107 \[ \det \begin{bmatrix} 4 & 6 \\ 4 & 9 \end{bmatrix} = 4 \cdot 9 - 4 \cdot 6 = 12. \] So this matrix is invertible, since \( 12 \neq 0 \).

108. \[ \det \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = 0 \] so this matrix is not invertible.

109. \[
\begin{align*}
\det \begin{bmatrix} 0 & 2 & -1 \\ 3 & 2 & 2 \\ -5 & 1 & 2 \end{bmatrix} &= 0 \det \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} - 2 \det \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} + (-1) \det \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \\
&= 0(2) - 2(16) + (-1)(13) \\
&= -45
\end{align*}
\]

110
\[
\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}
\]

To take the determinant of a \( 3 \times 3 \) matrix, we can expand along any row or column. Note we must take care of the sign of our expansion terms:

Look at this chessboard pattern:

\[
\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}
\]

Now let’s expand down the middle column. First take the entries of the middle column:

\[
\begin{align*}
&2 \\
&2 \\
&2
\end{align*}
\]

and switch their sign according to the chessboard pattern:

\[
\begin{align*}
&-2 \\
&+2 \\
&-2
\end{align*}
\]
Now each entry in our middle column lives on a row. If we strike out that row as well as the middle column we are left with a square $2 \times 2$ matrix. We do this for all three entries below:

$$
\begin{bmatrix}
* & [-]2 & * \\
1 & * & 3 \\
1 & * & 1
\end{bmatrix}
\begin{bmatrix}
1 & * & 5 \\
* & [+2] & * \\
1 & * & 1
\end{bmatrix}
\begin{bmatrix}
1 & * & 5 \\
* & [-2] & * \\
1 & * & 1
\end{bmatrix}
$$

Multiply each entry by the determinant of this (smaller) matrix:

$$
-2\det \begin{bmatrix}1 & 3 \\ 1 & 1\end{bmatrix} + 2\det \begin{bmatrix}1 & 5 \\ 1 & 1\end{bmatrix} - 2\det \begin{bmatrix}1 & 5 \\ 1 & 3\end{bmatrix}
$$

$$
= -2(1 - 3) + 2(1 - 5) - 2(3 - 5) = 0
$$

So the determinant is 0, and the matrix is not invertible.

Note that the first and second columns are linearly dependant, so we knew that the determinant was zero without computing it directly.

114. Since:

$$
\det \begin{bmatrix}0 & 1 & 2 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 4 & 8 & 15 \\ 3 & 7 & 15 & 30\end{bmatrix} = 1
$$

is not equal to zero, the matrix is invertible.

115. Let

$$
A = \begin{bmatrix}a & b \\ c & d\end{bmatrix}, \quad B = \begin{bmatrix}e & f \\ g & h\end{bmatrix}
$$

Then

$$
\det(AB) = \det \begin{bmatrix}ae + bg & af + bh \\ ce + dg & cf + dh\end{bmatrix}
$$

$$
= (ae + bg)(cf + dh) - (ce + dg)(af + bh)
$$

$$
= aecf + aedh + bgcf + bgdh - ceaf - cebh - dga - dbh
$$

$$
= aedh + bgcf - cebh - dga
$$

and

$$
\det(A) \det(B) = (ad - bc)(eh - gf) = adeh + bgcf - adgf - bc eh
$$

$$
= aedh + bgcf - cebh - adgf
$$

Thus $\det(AB) = \det(A) \det(B)$.

117. (a) The quadrilateral can be divided into two triangles, $\Delta_1$ with vertices $(0, 0)$, $(2, 1)$ and $(4, 1)$, and $\Delta_2$ with vertices $(0, 0)$, $(3, -3)$ and $(4, -1)$. Each triangle is half of a parallelogram.
Let 
\[ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \]

The first parallelogram is generated by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), so its area is
\[ \left| \det \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \right| = 6 \]

and the second parallelogram is generated by \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \), so its area is
\[ \left| \det \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix} \right| = 9 \]

Thus the area of \( R \) is \( 6/2 + 9/2 = 15/2 \).

(b) Since \( \det(A) = -4 \), the area of \( T(R) \) is 4 times the area of \( R \), which equals 30.

118. First we need to find the area of the triangle. Let:
\[ \mathbf{v}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \]

and,
\[ \mathbf{v}_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \]

Then the area of the triangle \( R \) is given by:
\[
\text{Area (} R \text{)} = \frac{1}{2} |\det[ \mathbf{v}_1 \mathbf{v}_2 ]| \\
= \frac{1}{2} \left| \det \begin{bmatrix} 3 & 4 \\ -2 & 4 \end{bmatrix} \right| \\
= 10.
\]

Finally, the area of \( T(R) \) is given by:
\[ |\det(T)| \ast \text{Area (} R \text{)} = 5 \ast 10 = 50. \]
119. (a) i. We have, by definition:

\[
[v]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2v_1 - v_2,
\]

so,

\[
[v]_S = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.
\]

ii. Similarly,

\[
[v]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = v_1 - 2v_2,
\]

so,

\[
[v]_S = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.
\]

i. Since:

\[
[v]_S = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

we have:

\[
[v]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

ii. Since:

\[
[v]_S = \begin{bmatrix} 9 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

we have:

\[
[v]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

121. (a) Since

\[
[T(v_1)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad [T(v_2)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad [T(v_3)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

the matrix for \( T \) with respect to \( B \) is

\[
B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]
Thus, if

\[
C = \begin{bmatrix}
1 & 2 & -1 \\
1 & 3 & 2 \\
1 & 0 & -6
\end{bmatrix}
\]

denotes the change of basis matrix

\[
A = CBC^{-1} = \begin{bmatrix}
-47 & 31 & 18 \\
-41 & 28 & 16 \\
-51 & 32 & 19
\end{bmatrix}
\]

(b) \( B^3 = I_3 \), so \( A^3 = (CBC^{-1})^3 = CB^3C^{-1} = CLI_3C^{-1} = CC^{-1} = I_3 \).

(c) \( A^{2000} = A^{1998+2} = A^{1998}A^2 = (A^3)^{666}A^2 = (I_3)^{666}A^2 = A^2 \), so

\[
A^{2000} = A^2 = \begin{bmatrix}
20 & -13 & -8 \\
-37 & 25 & 14 \\
116 & -77 & -45
\end{bmatrix}
\]

122. (a) We first need to find two linearly independent vectors orthogonal to \( v_1 \). These two vectors will automatically be a basis to the plane perpendicular to \( L \) (since a plane has dimension 2). Try:

\[
v_2 = \begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix},
\quad
v_2 = \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}.
\]

The set:

\[
\mathcal{B} = \{v_1, \ v_2, \ v_3\}
\]

is linearly independent since, for example,

\[
\det\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix} = 3 \neq 0.
\]

Thus, \( \mathcal{B} \) form a basis for \( R^3 \).

(b) From chapter 15, we have:

\[
P(v) = \frac{v \cdot v_1}{||v_1||^2}v_1.
\]

Observing the projection of \( v_1 \) onto the line \( L \) generated by itself is simply \( v_1 \), and that the projection of \( v_2 \) and \( v_3 \) onto the line perpendicular to both vectors.
is 0, we have:

First column of $B = B[v_1]_B = B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [P(v_1)]_B = [v_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

Second column of $B = B[v_2]_B = B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [P(v_2)]_B = [0]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

Second column of $B = B[v_3]_B = B \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [P(v_3)]_B = [0]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Thus,

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Let $C$ be the change of basis matrix:

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then,

$$C^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Thus, as in chapter 18,

$$A = CBC^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$