FINAL EXAM SOLUTIONS
You have 3 hours.
No notes, no books.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT
Good luck!

Name ________________________________
ID number _________________________

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

1. __________ (/50 points)

2. __________ (/50 points)

Signature: __________________________

3. __________ (/50 points)

Circle your TA’s name:

4. __________ (/50 points)

Kuan Ju Liu (2 and 6)

Robert Sussland (3 and 7)

5. __________ (/50 points)

Hunter Tart (4 and 8)

Bonus __________ (/20 points)

Alex Meadows (10)

Dana Rowland (11)

Total __________ (/250 points)

Circle your section meeting time:

11:00am 1:15pm 7pm
1. Let the function $f : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^2$ have components $f_1$ and $f_2$ as described by

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (x^2)^y \\ xy^2 \end{pmatrix}$$

(a) Note that the function $f$ is not defined at the origin; this is because the component $f_1$ is not defined there.

Is this discontinuity in $f_1$ removable? Justify your answer.

**Solution:** We compute limits of $f_1 = (x^2)^y$ as we approach the origin from different directions.

Along the $x$-axis, we have $y = 0$, so:

$$\lim_{x \to 0} f_1 = \lim_{x \to 0} (x^2)^0 = \lim_{x \to 0} 1 = 1$$

Along the $y$-axis, we have $x = 0$, so:

$$\lim_{y \to 0} f_1 = \lim_{y \to 0} (0^2)^y = \lim_{y \to 0} 0 = 0$$

Since these values are different, the limit must not exist. Therefore, the discontinuity in $f_1$ is not removable.

(b) Find the Jacobian matrix for the function $f$ at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

**Solution:**

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} (2y)x^{2y-1} \ln x^2(x^2)^y \\ y^2 \\ 2xy \end{pmatrix}$$

$$J_{f, \begin{pmatrix} 1 \\ 3 \end{pmatrix}} = \begin{pmatrix} 6 & 0 \\ 9 & 6 \end{pmatrix}$$
(c) In what (unit vector) direction $\vec{u}$ is the function $f_1$ increasing the fastest, at the point $(x, y) = (1, 3)$?

Solution: Since we know the first row of the Jacobian matrix is the gradient vector of $f_1$, we see immediately that

$$\nabla f \left( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

So, the direction (unit vector) in which the function is increasing the fastest is

$$\vec{u} = \frac{\nabla f}{\|\nabla f\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(d) What is $D_u f_1$ at the point $(x, y) = (1, 3)$, where $\vec{u}$ is the vector determined in part (c)?

Solution: As was shown in class, the directional derivative in the direction in which $f_1$ increases the fastest is equal to the length of the gradient of $f_1$. So,

$$D_u f_1 \left( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) = \left\| \nabla f \left( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) \right\| = \left\| \begin{pmatrix} 6 \\ 0 \end{pmatrix} \right\| = 6$$
2. Let the functions $f$ and $g$ be given by

\[
\begin{align*}
  f(t) &= \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} \\
  g \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= x_1^2 x_2^3 x_3
\end{align*}
\]

(a) Write down an equation for $\nabla g$.

**Solution:**

\[
\nabla g = \begin{pmatrix} 2x_1^2 x_2^3 x_3 \\ 3x_1^2 x_2^2 x_3 \\ x_1^2 x_2^3 \end{pmatrix}
\]

(b) Suppose that $f_1(t) = \sin t$, $f_2(t) = \cos t$, $f_3(t) = t^2$, and consider the composition $g \circ f$. Use the chain rule to find an expression (in terms of $t$) for

\[
\frac{dg}{dt}
\]

**Solution:** Since $g \circ f$ has only one input variable and one output variable, we have that

\[
J_{gof} = \left( \frac{dg}{dt} \right)
\]

And by the chain rule,

\[
J_{gof,t} = J_{g,f(t)}J_{f,t} = \nabla g(f(t)) \cdot f'(t) = \nabla g(f(t)) \cdot \begin{pmatrix} f'_1(t) \\ f'_2(t) \\ f'_3(t) \end{pmatrix} = \begin{pmatrix} 2x_1^2 x_2^3 x_3 \\ 3x_1^2 x_2^2 x_3 \\ x_1^2 x_2^3 \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ -\sin t \\ 2t \end{pmatrix}
\]

In this last expression of course, $x_i = f_i$, since this gradient is to be evaluated at $f(t)$.

\[
= 2x_1^3 x_2 x_3 \cos t - 3x_1^2 x_2^2 x_3 \sin t + 2x_1^2 x_2^3 t
\]

\[
= 2(\sin t)(\cos t)^3(t^2) \cos t - 3(\sin t)^2(\cos t)^2(t^2) \sin t + 2(\sin t)^2(\cos t)^3 t
\]

\[
= 2t^2 \sin t \cos^4 t - 3t^2 \sin^3 t \cos^2 t + 2t \sin^2 t \cos^3 t
\]
(c) Suppose instead that you do not have formulas for the components of \( f \); instead, you are given only that
\[
\begin{pmatrix}
2 \\
3 \\
0
\end{pmatrix}, \text{ and } \frac{dg}{dt}(0) = 5.
\]

Find the value of \( \frac{df_3}{dt}(0) \)

Solution:
\[
5 = \frac{dg}{dt}(0) = \nabla g(f(0)) \cdot \begin{pmatrix}
\frac{df_1}{dt}(0) \\
\frac{df_2}{dt}(0) \\
\frac{df_3}{dt}(0)
\end{pmatrix}
\]
\[
= \nabla g \begin{pmatrix}
2 \\
3 \\
0
\end{pmatrix} \cdot \begin{pmatrix}
\frac{df_1}{dt}(0) \\
\frac{df_2}{dt}(0) \\
\frac{df_3}{dt}(0)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
0 \\
0 \\
108
\end{pmatrix} \cdot \begin{pmatrix}
\frac{df_1}{dt}(0) \\
\frac{df_2}{dt}(0) \\
\frac{df_3}{dt}(0)
\end{pmatrix}
\]
\[
= 108 \frac{df_3}{dt}(0)
\]
So we conclude that
\[
\frac{df_3}{dt}(0) = \frac{5}{108}
\]
3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ have component functions $f_i : \mathbb{R}^n \to \mathbb{R}^1$.

(a) Suppose that at a point $\vec{a} \in \mathbb{R}^n$, the vectors $\{\nabla f_1, ..., \nabla f_n\}$ are dependent. Show that there must exist some non-zero vector $\vec{v}$ with

$$D_{f, \vec{a}}(\vec{v}) = \vec{0}$$

(Hint: Recall that the vectors $\{\nabla f_1, ..., \nabla f_n\}$ are the row vectors of the matrix $J_{f, \vec{a}}$.)

**Solution:** Since the dependent vectors $\{\nabla f_1, ..., \nabla f_n\}$ are the row vectors of the matrix $J_{f, \vec{a}}$, we conclude that the row space of $J_{f, \vec{a}}$ is of dimension at most $n - 1$.

Since the null space is the vector subspace perpendicular to the null space, we conclude that the dimension of the null space must be at least 1.

Thus, there is a non-zero vector $\vec{v}$ in the null space of $J_{f, \vec{a}}$, i.e.,

$$J_{f, \vec{a}}(\vec{v}) = \vec{0}$$

and so

$$D_{f, \vec{a}}(\vec{v}) = \vec{0}$$
(b) Use the result of part (a) to show that if the vectors

\[ \left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\} \]

are dependent at a point \( \overrightarrow{a} \in \mathbb{R}^n \), then we can draw the same conclusion – that there must exist some non-zero vector \( \overrightarrow{v} \) with

\[ D_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0} \]

(Hint: Recall the relationship between the dimensions of the row space and the column space of a matrix, and then use the result of part (a).)

Solution: The vectors

\[ \left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\} \]

are the column vectors of the matrix \( J_f \); so, if they are dependent, then the column space can have dimension at most \( n - 1 \). But the column space has the same dimension as the row space, so the row space must also have dimension at most \( n - 1 \).

Therefore our conclusion comes as a consequence of part (a).

(Alt: The dimension of the column space is at most \( n - 1 \), so by the Rank-Nullity Theorem, the dimension of the null space is at least 1. The conclusion again follows as in part (a).)
4. (a) Consider the function

\[ f(x, y) = \begin{pmatrix} x^2 - y^2 \\ x^2 + y^2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \]

Find and identify all critical points of the function \( h = \|f\|^2 \).

**Solution:**

\[ h = \|f\|^2 = (x^2 - y^2)^2 + (x^2 + y^2)^2 = 2x^4 + 2y^4 \]

\[ \nabla h = \begin{pmatrix} 8x^3 \\ 8y^3 \end{pmatrix} = 0 \quad \implies \quad x = y = 0 \]

So the origin is the only critical point.

\[ H = \begin{pmatrix} 24x^2 & 0 \\ 0 & 24y^2 \end{pmatrix} \quad \implies \quad H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \implies \quad \det H_0 = 0 \]

So, the second derivative test fails at this critical point.

However, we can clearly conclude that this critical point is a minimum, since the value of \( h \) there is 0, and clearly \( h \geq 0 \) for all points in \( \mathbb{R}^2 \), since both of the terms are even powers of real numbers.
(b) Consider the function

\[ f(x, y) = 5x^2 + y^2 + xy + 17x + y + 17 \]

Find and identify all critical points of \( f \).

**Solution:**

\[ \nabla f = \begin{pmatrix} 10x + y + 17 \\ 2y + x + 1 \end{pmatrix} = \vec{0} \quad \implies \]

\[ \begin{align*}
10x + y &= -17 \\
x + 2y &= -1
\end{align*} \]

\[ \implies 19x = -33 \]

\[ \implies x = \frac{-33}{19} \]

\[ \implies y = \frac{7}{19} \]

This is the only critical point.

\[ H = \begin{pmatrix} 10 & 1 \\ 1 & 2 \end{pmatrix} \quad \implies \quad \det H = 19 \]

Since both \( \det H \) and \( f_{xx} \) are positive, we conclude that this critical point is a minimum.
5. Consider the function

\[ f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + y + z \]

(a) Find the point which achieves the absolute minimum value of \( f \) on the surface \( x^2 + y^2 = z \).

**Solution:** Our restriction function \( g \) is \( g = x^2 + y^2 - z = 0 \). The gradients of \( f \) and \( g \) are then

\[ \nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \nabla g = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} \]

Since \( \nabla f \) must be a multiple of \( \nabla g \), and since the components of \( \nabla f \) are equal, we see that the same must be true of the components of \( \nabla g \). So, we conclude that

\[ x = y = -\frac{1}{2} \]

and therefore, using our \( g \) restriction,

\[ z = \frac{1}{2} \]
(b) Find the points which achieve the absolute minimum and maximum values of the function $f$ on the curve which is the intersection of the surfaces $x^2 + y^2 = z$ and $y + z = 1$.

Solution: We still have the same function $f$; this time we have two restriction functions $g_1 = x^2 + y^2 - z = 0$, and $g_2 = y + z - 1 = 0$. The relevant gradients are then

\[
\nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \nabla g_1 = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} \quad \nabla g_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

So we get three equations

\[
1 = \lambda_1(2x) + \lambda_2(0) \\
1 = \lambda_1(2y) + \lambda_2(1) \\
1 = \lambda_1(-1) + \lambda_2(1)
\]

From the last two of these equations, we conclude that $2y = 1$, so that

\[
y = -\frac{1}{2}
\]

Our $g_2$ restriction then tells us that

\[
z = \frac{3}{2}
\]

and then we conclude from the $g_1$ restriction that

\[
x = \pm \frac{\sqrt{5}}{2}
\]

So, the two points in question are

\[
\begin{pmatrix} \frac{\sqrt{5}}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{\sqrt{5}}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}
\]

Plugging into the function $f$, we conclude that the first of these is the maximum, and the second is the minimum.
**Bonus Question:** Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^3$ has components \( \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \), and that $D_{f,0}$ is the linear transformation which rotates vectors by an angle of 90° around the line spanned by \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \), in the direction that takes the z-axis towards the positive half of the x-axis.

Use this to calculate

\[ \frac{\partial f_2}{\partial z} \]

**Solution:**

\[ \frac{\partial f_2}{\partial z} = D_{e_3} f_2 = D_{f_2} (e_3) \]

This is of course just the second component of

\[ D_f (e_3) \]

Following the description of $D_f$ given in the statement of the problem, we see that $e_3$ is rotated to the vector

\[ D_f (e_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \]

So, we conclude that

\[ \frac{\partial f_2}{\partial z} = D_{f_2} (e_3) = (D_f (e_3))_2 = -\frac{1}{\sqrt{2}} \]