Quiz 6 Solutions

1a. (i) First check whether the \( n \)th term goes to 0:

\[
\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x \to \infty} \frac{\sin(x)}{x} = \lim_{y \to 0} \frac{\sin y}{y} = 1
\]

by l’Hospital’s Rule. Since the \( n \)th term does not have limit 0, the series diverges.

(ii) The series is a convergent geometric series, since

\[
\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n,n^3 \to \sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n,\]

with the absolute value of the ratio \(-\frac{9}{10}\) < 1. The series starts with \( n = 1 \), so the sum is

\[
\frac{-\frac{9}{10}}{1 - (-\frac{9}{10})} = -\frac{9}{19}.
\]

1b. Let \( a_n = \frac{n^3}{2^{n+1}} \), and apply the Ratio Test:

\[
\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{2^{n+2}} \cdot \frac{2^n}{n^3}\right| = \frac{1}{2} \left(1 + \frac{1}{n}\right)^3 |x| \to \frac{1}{2} |x|
\]

as \( n \to \infty \). So the power series converges for \( |x| < 2 \) and diverges for \( |x| > 2 \), and then the radius of convergence is 2. The Ratio Test is inconclusive on the boundary points -2 and 2, so we need to check those separately. For \( x = 2 \), the series diverges since it becomes \( \frac{1}{2} \sum_{n=1}^{\infty} n^3 \). And for \( x = -2 \), the series also diverges, since \( \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n n^3 \) diverges. So the interval of convergence is \((-2, 2)\).

2a. The series \( \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/n} \) is not absolutely convergent since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges (\( p \)-series with \( p = \frac{1}{2} \)). But by the Alternating Series Test, the series converges:

\[
\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}, \quad \text{and} \quad \frac{1}{\sqrt{n}} \to 0.
\]

Thus, the series is convergent but not absolutely convergent.

2b. We will use the Ratio Test (often a good idea when factorials are present):

\[
\left|\frac{(-1)^{n+1} e^{n+1} n!}{(n+1)! (-1)^n e^n}\right| = \frac{e}{n+1} \to 0 < 1,
\]

so the series \( \sum_{n=0}^{\infty} (-1)^n e^n \) is absolutely convergent.

2c. Note that the first term is 0 and the other terms are positive, so the series converging absolutely is the same thing as the series converging. Since \( \ln n < n \) and \( \frac{n}{n^2 - 2} \) is approximately \( \frac{n}{n^2} = \frac{1}{n} \) for large \( n \), we expect the series to converge. More precisely, \( \ln n < n \), so the original series converges if \( \sum_{n=2}^{\infty} \frac{n}{n^2 - 2} \) converges.

Also,

\[
\frac{n}{n^3 - 2} < \frac{2}{n^2}
\]

for \( n \geq 2 \) (as seen by cross multiplying), and \( \sum_{n=2}^{\infty} \frac{2}{n^2} \) converges, so by the Comparison Test, \( \sum_{n=2}^{\infty} \frac{n}{n^3 - 2} \) converges. Therefore, \( \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \) converges (absolutely).