MATH 42
SECOND SAMPLE MIDTERM #1

90 Minutes

NAME:

Section Number:

I agree to abide by the Honor Code.
Signature:

Instructions: Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. For each of the following integrals, either give an exact value, or else prove that it diverges.

(i) \[ \int_{-1}^{6} x \sqrt{x + 1} \, dx \]

(ii) \[ \int_{1}^{\infty} x e^x \, dx \]
(iii) \[ \int_3^6 \frac{dz}{z^2 + z - 2} \]

(iv) \[ \int_{-1}^1 \frac{1}{x^2} dx \]
2. Find the two points of intersection of the parabola $y = x(x - \pi)$ and the curve $y = \sin(x)$. Find the exact area between the two graphs.
3. A bowl is made in the shape you would obtain by rotating the curve
\[
y = \begin{cases} 
\frac{1}{5}x^2 & \text{for } 0 \leq x \leq 5 \\
2x - 5 & \text{for } 5 < x \leq 6 
\end{cases}
\]
around the y-axis. (Thus units of x and y are in inches.) Find the total capacity of the bowl in cubic inches.
4. Let $f(x)$ be a smooth function defined for all $x$. Set
\[ g(x) = \lim_{s \to 0} \frac{f(x + s) - f(x)}{s} \]
and
\[ h(x) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{g(x + \frac{i}{n})}{n}. \]
Write $h(x)$ in terms of $f(x)$ and explain your answer.
5. Suppose that $f(x)$ is a continuous, and $h(x)$ is differentiable and decreasing. Let

$$g(x) = \int_{h(x)}^{2} |f(w)| \, dw.$$  

Show that $g$ is non-decreasing.
6. For your 21st birthday, you get various gifts.
   (a) Your “plan for the future” grandmother gives you a continuous
       income stream at the rate of $(20+t)$ per year for 20 years. Here $t$
       is time in years starting right now. What is the present value of this gift.
       (Assume 5% interest.)

   (b) Your “crazy anthropologist” uncle gives you a rare ceremonial
       mask he uncovered on his last trip to Borneo. Since you only care
       about money, you decide to sell it. An expert tells you that its price $t$
       years from now will be approximately $700 + 40t + t^2$ dollars. When is
       the best time to sell? (Assume 5% interest.)
7. (a) Suppose that a function \( h(x) \) satisfies \( \int_1^3 h(x) \, dx = 6 \) and \( h(1) = -2 \) and \( h(3) = -7 \). Find \( \int_1^3 xh'(x) \, dx \).
(b) Suppose that a function $g(x)$ satisfies $\int_1^3 \frac{g(x)}{x^2} \, dx = 5$. Calculate $\int_1^3 g(3/x) \, dx$. 
8. Suppose that on a certain day a windmill generates electricity at $f(t)$ kilowatts, $t$ hours after noon. Let $F(t)$ be the total amount of energy (in kilowatt-hours) generated between noon and $t$ hours after noon. If $f(t) = \begin{cases} t^4e^{-t} & \text{for } 0 \leq t \leq 2 \\ 4(t/e)^2 & \text{for } t > 2 \end{cases}$, find $F'(3)$. 