MATH 42
SECOND SAMPLE MIDTERM #1

90 Minutes

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. For each of the following integrals, either give an exact value, or else prove that it diverges.

(i) \( \int_{-1}^{6} x\sqrt{x + 1} \, dx \)

Make the substitution \( u = x + 1 \). The limits of integration now become 0 to 7. We have \( \int_{0}^{7} (u - 1) \sqrt{u} \, du = \left( \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right) \bigg|_{0}^{7} \). This equals \( \frac{22\sqrt{7}}{15} \).

(ii) \( \int_{1}^{\infty} xe^x \, dx \)

First we compute \( \int_{1}^{a} xe^x \, dx \) using integration by parts. We get \( (xe^x - e^x)|_{1}^{a} = (1 - a)e^a \). But the limit as \( a \) goes to infinity of this expression diverges. Therefore, the improper integral diverges.

(iii) \( \int_{3}^{6} \frac{dz}{z^2 + z - 2} \)

We use partial fractions; \( \frac{1}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1} \) gives 1 = \( A(z-1) + B(z+2) \). It follows that \( A = -1/3 \) and \( B = 1/3 \). Thus, the original definite integral equals \( \frac{1}{3}(-\ln |z + 2| + \ln |z - 1|) \bigg|_{3}^{6} = \frac{1}{3}(-\ln 8 + \ln 5 + \ln 5 - \ln 2) = \frac{1}{3}\ln \frac{25}{16} \).

(iv) \( \int_{-1}^{1} \frac{1}{x^2} \, dx \)

This is an improper integral since the integrand is singular at zero. It can be rewritten as \( \lim_{a \to 0^-} \int_{-1}^{a} \frac{1}{x^2} \, dx + \lim_{b \to 0^+} \int_{b}^{1} \frac{1}{x^2} \, dx \). I claim that this is divergent. It is enough to show that one of the limits diverges (in fact, they both do). The second integral equals \( -\frac{1}{x} \bigg|_{b}^{1} = -1 + \frac{1}{b} \). As \( b \) approaches zero, this becomes unbounded.
2. Find the two points of intersection of the parabola $y = x(x - \pi)$ and the curve $y = \sin(x)$. Find the exact area between the two graphs.

The points of intersection are $(0, 0)$ and $(\pi, 0)$. The area between the graphs is given by the definite integral $\int_0^\pi \sin(x) - x^2 + \pi x \, dx$. By the first form of the fundamental theorem of calculus, we may compute this definite integral using an antiderivative of the integrand. It is easy to see that $F(x) = -\cos(x) - x^3/3 + \pi x^2/2$ is an antiderivative. The integral equals $F(x)|_0^\pi = (1 - \pi^3/3 + \pi^3/2) - (-1) = 2 + \pi^3/6$. 
3. A bowl is made in the shape you would obtain by rotating the curve

\[
y = \begin{cases} \\
\frac{1}{5}x^2 & \text{for } 0 \leq x \leq 5 \\
2x - 5 & \text{for } 5 < x \leq 6 \\
\end{cases}
\]

around the y-axis. (Thus units of x and y are in inches.) Find the total capacity of the bowl in cubic inches.

First, note that for \( y \leq 0 \leq 5 \), \( x = \sqrt{5y} \), and for \( 5 < y \leq 7 \), \( x = (y + 5)/2 \). Now slicing perpendicular to the y-axis, the slices have volume \( \pi x^2 \Delta y \). Thus, volume of bowl equals

\[
\int_{0}^{5} 5\pi y \, dy + \int_{5}^{7} \pi \left(\frac{y + 5}{2}\right)^2 \, dy.
\]

This equals

\[
5\pi \frac{y^2}{2} \bigg|_{0}^{5} + \pi \frac{(y + 5)^3}{3} \bigg|_{5}^{7},
\]

which is \( \frac{\pi y}{6} \) cubic inches.
4. Let $f(x)$ be a smooth function defined for all $x$. Set

$$g(x) = \lim_{w \to 0} \frac{f(x + w) - f(x)}{w}$$

and

$$h(x) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{g(x + \frac{i}{n})}{n}.$$ 

Write $h(x)$ in terms of $f(x)$ and explain your answer.

By definition of derivative, $g = f'$. By definition of definite integral, $h(x) = \int_{x}^{x+1} g(t) dt$. Now, using the FTC, $h(x) = f(x+1) - f(x)$. 
5. Suppose that \( f(x) \) is a continuous, and \( h(x) \) is differentiable and decreasing. Let

\[
g(x) = \int_{h(x)}^{2} |f(w)| \, dw.
\]

Show that \( g \) is non-decreasing.

Let \( s(x) = \int_{2}^{x} |f(w)| \, dw \). Then, \( g(x) = -s(h(x)) \). Thus, by the chain rule, \( g'(x) = -h'(x) s'(h(x)) \). By the Fundamental Theorem of Calculus, this equals \(-h'(x)|f(h(x))|\). Obviously, the absolute value is never negative. Also, since \( h \) is decreasing, \(-h'(x)\) is always positive. It follows that \( g'(x) \geq 0 \). Thus, \( g \) is non-decreasing.
6. For your 21st birthday, you get various gifts.
(a) Your “plan for the future” grandmother gives you a continuous income stream at the rate of $20 + t$ per year for 20 years. Here $t$ is time in years starting right now. What is the present value of this gift. (Assume 5% interest.)

The present value is given by the integral

$$\int_0^{20} (20 + t)e^{-0.05t} \, dt.$$  

We compute this using parts with $f = 20 + t$, $f' = 1$, $g' = e^{-0.05t}$, $g = -20e^{-0.05t}$. We get $(20 + t)(-20e^{-0.05t})\big|_0^{20} + 20 \int_0^{20} e^{-0.05t} \, dt = -800e^{-1} + 400 - 400e^{-0.05(20)} = 800 - 1200e^{-1}$ dollars.

(b) Your “crazy anthropologist” uncle gives you a rare ceremonial mask he uncovered on his last trip to Borneo. Since you only care about money, you decide to sell it. An expert tells you that its price $t$ years from now will be approximately $700 + 40t + t^2$ dollars. When is the best time to sell? (Assume 5% interest.)

The present value of selling in $t$ years is $(700 + 40t + t^2)e^{-0.05t}$. We want to know where this is maximum. The derivative is $-0.05(700 + 40t + t^2)e^{-0.05t} + (40 + 2t)e^{-0.05t}$ For this to equal zero, we must have $(700 + 40t + t^2) - 20(40 + 2t) = 0$. That is, $t^2 - 100 = 0$. Thus, the only critical point is $t = 10$. Moreover, it is easy to see that this is a maximum. Thus, it is best to sell in 10 years.
7. (a) Suppose that a function $h(x)$ satisfies $\int_1^3 h(x) \, dx = 6$ and $h(1) = -2$ and $h(3) = -7$. Find $\int_1^3 xh'(x) \, dx$.

Use integration by parts with $f = x$ and $g' = h'$. So, $f' = 1$ and $g = h$. Thus we have $\left. xh(x) \right|_1^3 - \int_1^3 h(x) \, dx$. Using the information given, this is $3h(3) - h(1) - 6 = -25$.

(b) Suppose that a function $g(x)$ satisfies $\int_1^3 \frac{g(x)}{x^2} \, dx = 5$. Calculate $\int_1^3 g(3/x) \, dx$.

Make the substitution $u = 3/x$. Then $du = -\frac{3}{u^2} \, dx$. This may be written as $-\frac{3}{u^2} \, du = dx$. Also, when $x = 1$, $u = 3$ and when $x = 3$, $u = 1$. Therefore, we get $-3 \int_1^3 \frac{g(u)}{u^2} \, du$. This is the same as $3 \int_1^3 \frac{g(u)}{u^2} \, du$. Using the information given, we see that this is $3(5) = 15$. 


8. Suppose that on a certain day a windmill generates electricity at \( f(t) \) kilowatts, \( t \) hours after noon. Let \( F(t) \) be the total amount of energy (in kilowatt-hours) generated between noon and \( t \) hours after noon. If \( f(t) = \begin{cases} t^4 e^{-t} & \text{for } 0 \leq t \leq 2 \\ 4(t/e)^2 & \text{for } t > 2 \end{cases} \), find \( F'(3) \).

We have \( F(t) = \int_0^t f(u) \, du \). Thus, by the Fundamental Theorem of Calculus, \( F'(3) = f(3) \). This equals \( 36/e^2 \) kilowatts.