MATH 42
SECOND SAMPLE FINAL EXAM

Three Hours

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. A tank contains 300 gallons of water and 100 gallons of pollutants (so it contains 400 gallons total). Fresh water is pumped into the tank at a rate of 2 gallons per minute and the well-stirred mixture leaves at the same rate.

(a) Write a differential equation for \( Q(t) \), the quantity (in gallons) of pollutants in the tank at time \( t \).

The rate of change of \( Q \) is the rate of pollutants in minus the rate out. Since fresh water is pumped in, the rate in is zero. The rate out is 2 \( \frac{Q(t)}{400} \). Thus,

\[
\frac{dQ}{dt} = -\frac{Q}{200}
\]

(b) How long does it take until there are only 10 gallons of pollutants in the tank?

Solving the equation from (a) gives \( Q(t) = Ce^{-t/200} \). Since \( Q(0) = 100, C = 100 \). Thus, to determine when 10 gallons of pollutants are left, we must solve \( 10 = 100e^{-t/200} \). This gives \( t = -200 \ln(1/10) = 200 \ln(10) \) minutes.
2. Determine whether each of the following series converges or diverges. If it diverges, explain how you know. If it converges, find the sum.

(a) \[ \sum_{n=1}^{\infty} \frac{\ln n}{n} \]

For \( n \geq 3 \), \( \ln n > 1 \). Thus, \( \frac{\ln n}{n} > \frac{1}{n} \). Since the harmonic series diverges, the given series diverges.

(b) \[ \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} \]

Note that \( \frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3} \). The given sum telescopes: the \( n \)th partial sum is

\[
(\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + \cdots + \left( \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{1}{3} - \frac{1}{n+3}
\]

Passing to the limit as \( n \to \infty \), the partial sums approach \( \frac{1}{3} \). Thus, the series converges to \( \frac{1}{3} \).

(c) \[ \sum_{n=1}^{\infty} \frac{4^{n+2}}{7^{n-1}} \]

We can rewrite this sum as \( \sum_{n=1}^{\infty} 64 \left( \frac{4}{7} \right)^{n-1} \). It is now obviously a convergent geometric series. The sum is \( 64 \frac{1}{1-\frac{4}{7}} = 64 \frac{7}{3} = 448/3 \).

(d) \[ \sum_{n=1}^{\infty} \sin(\pi/n) \]

I claim that this series diverges. To prove this, I will use the limit comparison test with the divergent series \( \sum \frac{1}{n} \). We get

\[ \lim_{n \to \infty} \frac{\sin(\pi/n)}{\pi/n} = \lim_{x \to 0} \frac{\sin(x)}{x} = 1. \]

Since this limit is finite and positive, the given series must diverge.
3. Just outside of Math42ville there is a lake where a unique species of fish lives (the only fish with expertise in calculus). Extensive research has shown that the population of fish is governed by the differential equation

\[
\frac{dP}{dt} = P - \frac{1}{10}P^2
\]

where \( P \) is number of fish in thousands and \( t \) is in years, and that there are currently 4000 fish in the lake. Since the residents of this town are deeply in debt (student loans) they decide to start fishing in the lake. Suppose that fish are removed at a continuous rate of \( D \) thousand fish per year. How large can \( D \) be so that the population of fish does not eventually die out? Carefully explain how you know.

If fish are removed at the rate of \( D \) thousand per year, the differential equation governing the population becomes \( \frac{dP}{dt} = P - \frac{1}{10}P^2 - D = -\frac{1}{10}(P^2 - 10P + 10D) \). The stable populations are obviously the solutions to \( P^2 - 10P + 10D = 0 \). Suppose that the solutions are \( A, B \) with \( A > B \). Then it is easy to see that for initial populations larger than \( B \), over time the population approaches the equilibrium \( P = A \), whereas initial populations less than \( B \) eventually die out. It follows that, given an initial population of 4000 fish, the population will survive exactly when \( B = (10 - \sqrt{100 - 40D})/2 \geq 4 \). In other words, \( \sqrt{100 - 40D} \geq 2 \). So, \( D \geq \frac{96}{40} = 2.4 \). This shows that as long as fish are not removed at a rate greater than 2400 per year, the population will not die out.
4. Having completed his training in the Dagobah system, the young Jedi returns to his home planet of Tatooine to settle old debts with a vile 4000-pound gangster. Exactly 2 years ago the gangster lent him 20,000 credits (the local currency). As repayment the Jedi must pay a continuous income stream of $kt$ credits per year lasting forever. ($t =$time in years.) To make the deal fair, what should $k$ be? Assume that interest on Tatooine is continuously compounded at 5%.

Say $t = 0$ is present time - that is, when the Jedi arrives on Tatooine. At this time, he owes the gangster 20,000$e^{(1/20)(2)} = 20,000e^{0.1}$ credits. To be fair, this must equal the present value of his repayment,

$$\int_0^\infty kte^{-t/20} dt.$$  

Using integration by parts, this equals

$$\lim_{a \to \infty} kt(-20e^{-t/20})|_0^a + 20k \int_0^\infty e^{-t/20} dt$$

This equals

$$-20k \lim_{a \to \infty} ae^{-a/20} - 400k \lim_{b \to \infty} e^{-b/20}|_0^b$$

So, the present value of the repayment is

$$-400k(\lim_{b \to \infty} e^{-b/20} - 1) = 400k.$$  

Thus, to be fair $20,000e^{0.1} = 400k$. This gives $k = 50e^{0.1}$. 
5. A rancher in Nevada leaves water for her cattle in a trough which is 9 feet long and has ends in the shape of isosceles triangles 1 foot wide and 2 feet high. Assume that water evaporates at a rate which is proportional to the area of water that is exposed (that is, the area of the top of the water). On a certain day, the trough is completely filled in the morning, but no cows come by and drink. After three hours in the sun, \( 23/16 \) cubic feet of water is lost to evaporation. How much is lost after 12 hours?

Let \( w \) be the width of the top of the water, \( A \) the area of the top of the water, and \( V \) the volume of the water. Obviously, \( A = 9w \). Also, by similar triangles, the height of the water is \( 2w \). It follows that \( V = 9w^2 \). We can relate \( A, V \) as follows. \( w = \sqrt{V}/3 \). Thus, \( A = 9w = 3\sqrt{V} \).

Now, we are given that, for some constant \( k \), \( \frac{dV}{dt} = kA \). From above, this is the same as

\[
\frac{dV}{dt} = 3k\sqrt{V}.
\]

Separating variables,

\[
\int \frac{dV}{\sqrt{V}} = \int 3k \, dt.
\]

So, \( 2\sqrt{V} = 3kt + C \) or

\[
V = (\frac{3}{2}kt + \frac{C}{2})^2.
\]

Since the trough is initially full, \( V(0) = 9 \). This implies that \( C = 6 \). Also, \( V(3) = 9 - \frac{23}{16} = \frac{121}{16} \). This allows us to find \( k \). We have \( \frac{11}{4} = \sqrt{\frac{121}{16}} = \frac{9}{2}k + 3 \). This implies that \( k = -1/18 \). Thus,

\[
V(t) = (\frac{-t}{12} + 3)^2
\]

After 12 hours, the volume of water is \( V(12) = 4 \). So, after 12 hours we have lost \( 9 - 4 = 5 \) cubic feet.
6. Compute the following indefinite integrals.

(a) \[ \int \frac{1}{x^2 + x - 2} \, dx \]

Observe that \( x^2 + x - 2 = (x + 2)(x - 1) \). Thus, we can use partial fractions to write \( \frac{1}{x^2 + x - 2} = \frac{A}{x + 2} + \frac{B}{x - 1} \). We get 1 = \( A(x - 1) + B(x + 2) \). It follows that \( A = -\frac{1}{3} \) and \( B = \frac{1}{3} \). The integral becomes 
\[ -\frac{1}{3} \int \frac{1}{x + 2} \, dx + \frac{1}{3} \int \frac{1}{x - 1} \, dx = -\frac{1}{3} \ln |x + 2| + \frac{1}{3} \ln |x - 1| + C. \]

(b) \[ \int (x + 1)\sqrt{x - 1} \, dx \]

Make the substitution \( u = x - 1 \). Then \( du = dx \) and \( x = u + 1 \). This gives \( f(u + 2)\sqrt{u} \, du = \frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} + C \). This equals \( \frac{2}{5} (x - 1)^{5/2} + \frac{4}{3} (x - 1)^{3/2} + C \).

(c) \[ \int \frac{1 - x}{x^2 - 2x + 3} \, dx \]

Make the substitution \( u = x^2 - 2x + 3 \). Then \( du = 2(x - 1) \, dx \) or \( -\frac{1}{2} \, dx = (1 - x) \, dx \). This gives \( -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C \). This equals \( -\frac{1}{2} \ln |x^2 - 2x + 3| + C \).

(d) \[ \int \frac{\ln(x)}{x^2} \, dx \]

Use integration by parts: \( f = \ln x \), \( g' = \frac{1}{x} \), \( f' = \frac{1}{x} \), \( g = -\frac{1}{x} \). This gives \( -\ln x + \int \frac{1}{x^2} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C \).
7. Let \( f(x) = \sec x \). Calculate \( f^{(4)}(0) \) by finding the first few terms of the Maclaurin series for \( \sec x = \frac{1}{\cos x} \) using the known Maclaurin series for \( \cos x \).

\[
\begin{align*}
\cos(x) &= 1 - x^2/2 + x^4/24 - \ldots. \\
\sec x &= 1/\cos(x) = 1 + x^2/2 + 5x^4/24 + \ldots.
\end{align*}
\]

So, using long division, \( \sec x = 1/\cos(x) = 1 + x^2/2 + 5x^4/24 + \ldots \). On the other hand, the \( x^4 \) term of the Maclaurin series for \( \sec(x) \) is \( f^{(4)}(0)x^4/24 \). It follows that \( f^{(4)}(0) = 5 \).
8. Suppose that a ball is dropped from height $H$ feet.
(a) Show that it takes $\frac{1}{4}\sqrt{H}$ seconds for the ball to reach the ground.
(Recall that gravity provides a downward acceleration of 32 feet per second squared.)

| Let $a, v, p$ be the acceleration, velocity and position of the ball, respectively. ($p = 0$ on the ground.) Since, $a = -32$ and the initial velocity is zero, $v(t) = -32t$. Also, $p(t) = H - 16t^2$. The ball hits the ground when $p(t) = 0$, that is, when $16t^2 = H$. We get $t = H^{1/2}/4$. |

(b) Suppose that the ball is dropped from a height of 10 feet, and that each time the ball bounces, it returns to a point $4/5$ as high as the previous bounce. Show that the ball travels a total of 90 feet.

| The ball travels 10 feet in reaching the ground, then it goes up to a height of $10(\frac{4}{5})$, then to $10(\frac{4}{5})^2$ and so on. Taking into consideration that it travels both up and down, the total distance travelled is $10 + 20(\frac{4}{5}) + 20(\frac{4}{5})^2 + 20(\frac{4}{5})^3 + \ldots$. This equals $-10 + \sum_{n=1}^{\infty} 20(\frac{4}{5})^{n-1} = -10 + 20 \cdot \frac{1}{1-\frac{4}{5}} = 90$. |

(c) Using parts (a) and (b), calculate how long it takes for the ball to stop bouncing. [In particular, we see that, although the ball bounces infinitely many times, it takes only finite time to do it.]

| By part (a), the ball takes $\frac{1}{4}\sqrt{10}$ seconds to reach the floor initially. On the $n$th bounce, it takes $\frac{1}{4}\sqrt{10}(\frac{4}{5})^n$ seconds. Taking into consideration that it travels both up and down (it’s easy to see that it takes the same amount of time to go up as to come down), the total time it takes is $\frac{1}{4}\sqrt{10} + \sum_{n=1}^{\infty} \frac{1}{2} \sqrt{10}(\sqrt{\frac{4}{5}})^n$ |

| This equals $\frac{1}{4}\sqrt{10} + \frac{1}{2} \sqrt{10} \sqrt{\frac{4}{5} - \frac{1}{\sqrt{\frac{4}{5}}}}$ |
9. Evaluate

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{(1 + \frac{i}{n}) \ln(1 + \frac{i}{n})}{n}$$

by interpreting it as a definite integral and then using the fundamental theorem of calculus.

By definition of definite integral, this limit equals \( \int_{1}^{2} x \ln x \, dx \).
This equals \(( \frac{x^2}{2} \ln x - \frac{x^2}{4} )\bigg|_{1}^{2} = 2 \ln(2) - \frac{3}{4} \).
10. (a) Use induction to prove that \( \int_{0}^{\infty} t^n e^{-t} \, dt = n! \).

First we prove the formula when \( n = 1 \). Using integration by parts

\[
\int_{0}^{\infty} t e^{-t} \, dt = \lim_{a \to \infty} -te^{-t} \big|_{0}^{a} + \int_{0}^{\infty} e^{-t} \, dt \\
= \lim_{b \to \infty} (-e^{-t}) \big|_{0}^{b} = 1
\]

We now assume that the formula is true for \( n = k \) and prove it for \( n = k + 1 \). Again using parts,

\[
\int_{0}^{\infty} t^{k+1} e^{-t} \, dt = \lim_{a \to \infty} -t^{k+1} e^{-t} \big|_{0}^{a} + (k + 1) \int_{0}^{\infty} t^k e^{-t} \, dt \\
= (k + 1)k! = (k + 1)!
\]

as claimed.

(b) Using the Maclaurin series for \( \sin x \) and part (a), prove that

\[
\int_{0}^{\infty} e^{-t} \sin(xt) \, dt = \frac{x}{1 + x^2}
\]

for \( |x| < 1 \).

\[
\int_{0}^{\infty} e^{-t} \sin(xt) \, dt = \int_{0}^{\infty} e^{-t} \sum_{n=0}^{\infty} \left( \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \, dt \\
= \sum_{n=0}^{\infty} \left( \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \int_{0}^{\infty} e^{-t} t^{2n+1} \, dt \\
= \sum_{n=0}^{\infty} \left( \frac{(-1)^n x^{2n+1}}{(2n+1)!} (2n + 1)! \right)
\]

The last step used part (a). We have now shown that

\[
\int_{0}^{\infty} e^{-t} \sin(xt) \, dt = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \\
= x \sum_{n=0}^{\infty} (-x^2)^n = \frac{x}{1 + x^2}
\]

The last step is just summing the geometric series.
11. (a) Find the general solution to $y'' + 6y' + 25y = 0$.

Observe that $r^2 + 64 + 25$ has roots

$$\frac{-6 \pm \sqrt{36 - 100}}{2} = -3 \pm 4i.$$ 

Thus, the general solution is

$$y(t) = C_1 e^{-3t} \cos(4t) + C_2 e^{-3t} \sin(4t).$$

This could also be written as $A e^{-3t} \sin(4t + \varphi)$.

(b) Find the solution to $y'' - 3y' + 2y = 0$ when $y(0) = 0$ and $y'(0) = 3$.

Since $r^2 - 3r + 2 = (r - 2)(r - 1)$, the general solution is $y(t) = C_1 e^{2t} + C_2 e^t$. Since $y(0) = 0$, $C_1 + C_2 = 0$. Also, $y'(0) = 3$ implies that $2C_1 + C_2 = 3$. Thus, $C_1 = 3$ and $C_2 = -3$. Therefore, the solution is

$$y(t) = 3e^{2t} - 3e^t.$$

(c) Find the solution to $y'' = -4y$ when $y(\pi/3) = 1$ and $y(\pi/4) = 0$.

This equation corresponds to simple harmonic motion. The general solution is $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$. But, $0 = y(\pi/4) = C_2$. It follows that, $1 = y(\pi/3) = -C_1/2$. Therefore, the solution is

$$y(t) = -2 \cos(2t).$$

(d) Find the solution to $y'' + 6y' - 7y = 0$ when $y(0) = 0$ and $y'(0) = 4$.

Since $r^2 + 6r - 7 = (r - 1)(r + 7)$, the general solution is $y(t) = C_1 e^t + C_2 e^{-7t}$. In addition, $0 = y(0) = C_1 + C_2$ and $4 = y'(0) = C_1 - 7C_1$. It follows that $C_2 = \frac{1}{2}$ and $C_1 = -\frac{1}{2}$. Thus,

$$y(t) = \frac{1}{2} e^t - \frac{1}{2} e^{-7t}.$$
12. Suppose that $f$ is a continuous function. Let
\[
g(x) = \frac{d}{dx} \int_0^x f(t) \, dt.
\]
Suppose that both $f$ and $g$ have convergent Taylor series, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Prove that
\[
b_n = \begin{cases} 
0 & \text{n even} \\
2a_{(n-1)/2} & \text{n odd}
\end{cases}
\]

By the fundamental theorem of calculus and the chain rule, $g(x) = 2xf(x^2)$. Thus,
\[
\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} 2a_n x^{2n+1}.
\]
Since only odd powers of $x$ appear on the right side, the same must be true on the left side. That is, $b_n = 0$ if $n$ is even. Finally, if $k$ is an odd number, the $x^k$ term on the right side occurs when $n = (k-1)/2$. It follows that, for $n$ odd, $b_n = a_{(n-1)/2}$.
13. After a climbing accident Dave is dangling at the end of a rope, 40 feet below a rock ledge. You and Josh are on the ledge and try to pull him up. You pull him up half way, before letting Josh take over. How much mechanical work have you done? (Note that Dave and his equipment weighs 200 pounds and the rope weighs 1/4 pound per foot.)

You lift Dave 20 feet. The work required to do this is $(20)(200) = 4000$ ft-lbs. You also lift the lower half of the rope 20 feet. This half of the rope weighs $20(1/4) = 5$ lbs. Thus, the work to lift the lower half of the rope is $(20)(5) = 100$ ft-lbs. Since you lift different parts of the upper half of the rope different distances, we must use an integral to compute the work required. We let $x$ represent the distance down from the ledge. Then, the work required is

$$\int_0^{20} x \frac{1}{4} \, dx = \frac{x^2}{8} \bigg|_0^{20} = 25 \text{ ft-lbs.}$$

All together, you do 4125 ft-lbs of work.
14. In many coastal areas, pigeons and sea gulls compete for the same resources. Let \( P(t) \) and \( S(t) \) be the number of pigeons and gulls (in hundreds) in one area. These populations satisfy the following system of differential equations.

\[
\frac{dP}{dt} = 6P - \frac{2}{5}P^2 - PS
\]
\[
\frac{dS}{dt} = 10S - S^2 - 2PS
\]

Analyze the phase plane for this system of equations (for \( P, S \geq 0 \), of course) showing the nullclines and equilibrium points, and sketch the direction of the trajectories in each region. Also, using this information, determine what happens in the long run if we start with 400 pigeons and 400 gulls.

We focus on the \( P - S \) plane. The nullclines occur where the tangents are either vertical or horizontal. Horizontal tangents occur when \( \frac{dS}{dt} = 0 \). This occurs on the lines \( S = 0 \) and \( 10 - S - 2P = 0 \). Similarly, vertical tangents occur along the lines \( P = 0 \) and \( 6 - \frac{2}{5}P - S = 0 \). We have indicated this on the diagram below. Observe that the equilibrium points are \( (0, 0), (15, 0), (0, 10) \) and \( (\frac{5}{2}, 5) \).

In each of the four regions of the phase plane, we have indicated the signs of the derivatives, and hence the general direction of the trajectories. From that it is obvious that a trajectory beginning at \( (4, 4) \) will eventually move toward the equilibrium \( (15, 0) \). In other words, starting with 400 pigeons and 400 gulls, the populations move toward 1,500 pigeons and no gulls – the gulls are driven out!