MATH 42
FIRST SAMPLE FINAL EXAM

Three Hours

NAME:

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.
Signature:

SOLUTIONS

Instructions: Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. Find the area between the curves $x + y^2 = 2$ and $x + y = 0$.

First observe that the curves intersect at the points $(-2, 2)$ and $(1, -1)$. Slicing the region parallel to the $x$-axis leads to the integral

$$
\int_{-1}^{2} [(2 - y)^2 + y] \, dy = \left(2y - \frac{y^3}{3} + \frac{y^2}{2}\right) \bigg|_{-1}^{2} = \frac{9}{2}.
$$
2. Using what you know about the Maclaurin series for \( \sin(x) \), find the Maclaurin series for \( g(x) = x \sin(x^2) \). Using this, find \( g^{(7)}(0) \) and \( g^{(13)}(0) \). (Do not differentiate!)

Since

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},
\]

\[
x \sin(x^2) = x \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!}.
\]

The coefficient of \( x^7 \) in this series is \( \frac{1}{3!} \). On the other hand, the coefficient in the Maclaurin series for \( g \) is \( \frac{g^{(7)}(0)}{7!} \). Therefore,

\[
g^{(7)}(0) = -\frac{7!}{3!} = -840.
\]

Next, observe that the coefficient of \( x^{13} \) in our series is zero. It follows that \( g^{(13)}(0) = 0 \).
3. A storage tank has the shape of the lower half of a sphere of radius 4. It is buried 3 feet under ground. (That is, the TOP of the tank is 3 feet below ground level.) Suppose that the tank is filled with a liquid which weighs $W$ pounds per cubic foot. How much work does it take to pump all the liquid out? (Answer in terms of $W$.)

We will use the variable $x$ where $x = 0$ at the top of the tank and $x = 4$ at the bottom of the tank. By the Pythagorean identity, a cross section at level $x$ has radius $\sqrt{16 - x^2}$. Therefore, a slice of thickness $\Delta x$ has volume approximately $\pi (16 - x^2) \Delta x$. The weight of this slice is approximately $W \pi (16 - x^2) \Delta x$. Also notice that to pump this liquid out, we need to lift it approximately $x + 3$ feet. Adding up the work required over all horizontal slices gives

$$\int_0^4 W \pi (16 - x^2)(3 + x) \, dx.$$ 

Since $(16 - x^2)(3 + x) = 48 + 16x - 3x^2 - x^3$, the work required is

$$W \pi \left[ 48x + 8x^2 - x^3 - \frac{x^4}{4} \right]_0^4 = 192W \pi$$

So, it takes $192W \pi$ foot-pounds of work to pump the liquid out.
4. Suppose that the differentiable function $f(x)$ satisfies

$$f(1) = -2, \quad \int_{1}^{4} f(x) \, dx = 3 \quad \text{and} \quad \int_{0}^{1} x f'(x) \, dx = -7.$$ 

Show that there is a $c$ in $[0, 4]$ so that $f(c) = 2$. (Hint: use the Mean Value Theorem for the interval $[0, 4]$.)

Using integration by parts,

$$\int_{0}^{1} x f'(x) \, dx = x f(x) \bigg|_{0}^{1} - \int_{0}^{1} f(x) \, dx = f(1) - \int_{0}^{1} f(x) \, dx.$$

Therefore,

$$\int_{0}^{1} f(x) \, dx = -2 + 7 = 5.$$ 

Next,

$$\int_{0}^{4} f(x) \, dx = \int_{0}^{1} f(x) \, dx + \int_{1}^{4} f(x) \, dx = 5 + 3 = 8.$$ 

Finally, by applying the Mean Value Theorem to $f$ on the interval $[0, 4]$, we know that there exists a $c$ in $[0, 4]$ so that

$$f(c) = \frac{1}{4-0} \int_{0}^{4} f(x) \, dx = \frac{8}{4} = 2.$$
5. Suppose that a fresh water aquarium that holds 1000L of water leaks at a rate of 2 L/day. The owner wants to keep the aquarium full, but since he lives near the ocean, he can only add salt water to the tank. Unfortunately, the fish in the aquarium will die if the salinity (concentration of salt in the water) reaches 0.05 kg/L. Suppose the salt water that the owner adds has a salinity of 0.1 kg/L, and that he adds it at a rate of 2L per day.

Do the fish eventually die? If so, when?

Since in the long run the salt approaches 0.1 kg/L, the fish eventually die. To determine when, let $S(t)$ be the amount of salt in the tank after $t$ days. We know $S(0) = 0$. Also,

$$\frac{dS}{dt} = 0.2 - \frac{2S}{1000} = -\frac{1}{500}(S - 100).$$

Separating variables,

$$\int \frac{dS}{S - 100} = \int -\frac{dt}{500}.$$

Thus, $\ln |S - 100| = -\frac{1}{500}t + C$. In other words,

$$S(t) = 100 + Ae^{-t/500}$$

for some constant $A$. Since $S(0) = 0$, $A = -100$. The fish die when $S(t) = (0.05)(1000) = 50$ kg. We must solve

$$50 = 100(1 - e^{-t/500}).$$

This implies

$$e^{-t/500} = \frac{1}{2}$$

so

$$t = -500 \ln(1/2).$$

In other words, the fish will die in 500 ln 2 days.
6. Compute the following indefinite integrals.

(a) \( \int \frac{1}{x \ln x} \, dx \)

Let \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \) and the integral becomes
\[
\int \frac{du}{u} = \ln |u| + C.
\]
Thus, the integral equals \( \ln |\ln x| + C \).

(b) \( \int \frac{2x + 1}{x + 2 + x^2} \, dx \)

Let \( u = x + 2 + x^2 \). Then \( du = (2x + 1) \, dx \) and the integral becomes
\[
\int \frac{du}{u} = \ln |u| + C.
\]
Thus, the integral equals \( \ln |x + 2 + x^2| + C \).
7. (a) The series \( \sum_{n=1}^{\infty} \frac{1}{(3-2n)(5-2n)} \) has partial sums \( s_n = \frac{n}{9-6n} \).

(You may assume this – you do not need to prove it.) Does the series converge? If so, to what value? If not, why not?

Observe that

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n}{9-6n} = \lim_{n \to \infty} \frac{1}{9/n - 6} = -\frac{1}{6}
\]

Since the sequence of partial sums converges, so does the series. The series converges to \(-\frac{1}{6}\).

(b) Find the interval of convergence for the power series \( \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n+1} \).

Observe that

\[
\lim_{n \to \infty} \left| \frac{2^{n+1} x^{n+1}/(n+2)}{2^n x^n/(n+1)} \right| = 2|x| \lim_{n \to \infty} \frac{n+1}{n+2} = 2|x|.
\]

Therefore, by the ratio test, the radius of convergence is \( \frac{1}{2} \).

Now let’s check the endpoints. When \( x = \frac{1}{2} \), we get the series \( \sum (-1)^n \frac{1}{n+1} \) which converges (alternating harmonic series). On the other hand, when \( x = -\frac{1}{2} \), the series becomes \( \sum \frac{1}{n+1} \) which diverges (harmonic series). Therefore, the interval of convergence is \( (-\frac{1}{2}, \frac{1}{2}] \).
8. While attempting to foil a diabolical scheme, Clark is captured by Dr. Evil’s henchmen. They tie bricks to his feet and throw him into a 16 foot high cylindrical tank filled with water. The tank has radius 5 feet.

When the henchmen leave, Clark is able to punch a hole in the bottom of the tank with his feet, allowing water to drain out at a rate proportional to the square-root of the depth of the water. He notes that after 2 minutes, the water level is only 9 feet.

Given that Clark’s nose is 6 feet above the bottom of the tank, how long will he have to hold his breath until the water level is low enough for him to breath?

Let $L(t)$ be the water level in the tank with $t$ in minutes. We know $L(0) = 16$ and $L(2) = 9$. Since $L$ is proportional to the volume of water in the tank, the rate of change of $L$ with respect to time is proportional to the square-root of $L$. That is,

$$\frac{dL}{dt} = -k\sqrt{L}.$$ 

Separating variables gives

$$\int \frac{dL}{\sqrt{L}} = \int -k\, dt.$$ 

Thus,

$$2\sqrt{L} = -kt + C$$

for some constant $C$. Since $L(0) = 16$, $C = 8$. Also, since $L(2) = 9$, $6 = -2k + 8$. Hence, $k = 1$. We must find $t$ so that $L(t) = 6$. That is, $2\sqrt{6} = -t + 8$. Thus, Clark must hold his breath for

$$8 - 2\sqrt{6}$$

minutes.
9. Compute the following indefinite integrals.
   (a) $\int \tan \theta \sec^3 \theta \, d\theta$

   Let $u = \sec \theta$. Then $du = \sec \theta \tan \theta \, d\theta$ and the integral becomes $\int u^2 \, du = \frac{u^3}{3} + C$. Therefore, the indefinite integral equals $\frac{\sec^3 \theta}{3} + C$.

(b) $\int (u + \sin u)e^{-u} \, du$

   First, using integration by parts
   \[
   \int ue^{-u} \, du = -ue^{-u} + \int e^{-u} \, du = -ue^{-u} - e^{-u} + C.
   \]

   Next, using integration by parts twice,
   \[
   \int e^{-u} \sin u \, du = -e^{-u} \sin u + \int e^{-u} \cos u \, du
   \]
   \[
   = -e^{-u} \sin u - e^{-u} \cos u - \int e^{-u} \sin u \, du
   \]

   Therefore,
   \[
   \int e^{-u} \sin u \, du = -\frac{e^{-u}}{2} (\sin u + \cos u) + C.
   \]

   Therefore
   \[
   \int (u + \sin u)e^{-u} \, du = -ue^{-u} - e^{-u} - \frac{e^{-u}}{2} (\sin u + \cos u) + C.
   \]
10. Suppose that $f$ is a differentiable function which satisfies
\[ \int_0^x f(t) \, dt = 1 + \frac{1}{f(x)} \]
for all $x \geq 0$. Find an explicit formula for $f(x)$.

Differentiating both sides of this equation with respect to $x$
and using the fundamental theorem of calculus gives
\[ f = -\frac{1}{f^2} \frac{df}{dx}. \]
Separating variables gives
\[ \int -f^{-3} \, df = \int dx. \]
Therefore $\frac{1}{2} f^{-2} = x + C$ and so
\[ f(x) = \pm \frac{1}{\sqrt{2x + C}}. \]
Finally, notice that in the original formula, when $x = 0$,
$f(0) = -1$. It follows that
\[ f(x) = -\frac{1}{\sqrt{2x + 1}}. \]
11. A sequence \( \{a_n\} \) is given by \( a_1 = \frac{3}{4} \) and
\[
a_{n+1} = 1 - \sqrt{1 - a_n}
\]
for \( n > 1 \). You may assume without proof that \( 0 < a_n < 1 \) for all \( n \).
(a) Show that \( \{a_n\} \) is monotonic.

It is enough to show that \( a_{n+1} < a_n \). I will do this by induction. If \( n = 1 \), we have \( a_2 = 1 - \sqrt{1/4} = \frac{1}{2} < a_1 \). Next, assume that \( a_{k+1} < a_k \). I will prove that \( a_{k+2} < a_{k+1} \). We have
\[
a_{k+2} - a_{k+1} = 1 - \sqrt{1 - a_{k+1}} - (1 - \sqrt{1 - a_k}) = \sqrt{1 - a_k} - \sqrt{1 - a_{k+1}}
\]
Since \( a_{k+1} < a_k \), this last expression is negative. It follows by induction that \( a_{n+1} < a_n \) for all \( n \).

(b) Explain how you know this sequence has a limit, and find the limit.

Since bounded, monotone sequences have limits, this sequence has a limit. Call this limit \( A \). Then
\[
\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} 1 - \sqrt{1 - a_n}
\]
gives \( A = 1 - \sqrt{1 - A} \). Thus, \((1 - A)^2 = 1 - A \). So, \( A = 0 \) or \( A = 1 \). But since the sequence is decreasing, it can’t converge to 1. Therefore, the sequence converges to 0.

(c) Determine whether \( \sum_{n=1}^{\infty} a_n \) converges or diverges. (Hint: consider ratio test.)

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1 - \sqrt{1 - a_n}}{a_n}
\]
Multiplying the numerator and denominator by \( 1 + \sqrt{1 - a_n} \) gives
\[
\lim_{n \to \infty} \frac{a_n}{a_n(1 + \sqrt{1 - a_n})} = \lim_{n \to \infty} \frac{1}{1 + \sqrt{1 - a_n}} = \frac{1}{2}
\]
Therefore, by the ratio test, the series converges.
12. Consider the following system of first order differential equations.
\[
\begin{align*}
\frac{dx}{dt} &= x - y \\
\frac{dy}{dt} &= x + y
\end{align*}
\]
(a) Analyze the phase plane for this system showing nullclines, equilibrium points and the direction of trajectories in each region.

(b) Find the solution to the system of equations when \(x(0) = 1\) and \(y(0) = 0\) by first converting the system to a single second order equation.

\[
y'' = x' + y' = x - y + y' \]

Using \(x = y' - y\), we get \(y'' = y' - y - y + y'\). In other words,
\[
y'' - 2y' + 2y = 0.
\]
The roots of \(r^2 - 2r + 2 = 0\) are \(1 \pm i\) so
\[
y(t) = e^t (A \cos t + B \sin t)
\]
for some \(A, B\). Now, \(y(0) = 0\) and \(y'(0) = x(0) + y(0) = 1\). It follows that \(A = 0\) and \(B = 1\). Therefore,
\[
y(t) = e^t \sin t
\]
Also, \(x(t) = y' - y = e^t \sin t + e^t \cos t - e^t \sin t\). Thus,
\[
x(t) = e^t \cos t
\]
13. Suppose that the population of skunks in Math42ville is governed by the logistic equation. The population is in equilibrium at 10 thousand skunks. As an experiment, residents decide to start killing them off at the rate of 3 thousand skunks per year. This causes the population to decline until it approaches a new equilibrium of 9 thousand skunks. Based on this data, at what rate would we have to kill skunks in order to eventually bring the population down to a stable 6 thousand skunks?

\[
\frac{dp}{dt} = -aP(P - 10)
\]

for some constant \(a\). When we are killing them at 3 thousand a year,

\[
\frac{dP}{dt} = -aP(P - 10) - 3 = -a(P^2 - 10P + 3a).
\]

Since \(P = 9\) is an equilibrium, \(9^2 - 10(9) + 3a = 0\). It follows that \(a = 3\).

Now suppose we kill the skunks at the rate of \(D\) thousand a year:

\[
\frac{dP}{dt} = -3(P^2 - 10P + 3D)
\]

In order to get an equilibrium at \(P = 6\), \(6^2 - 10(6) + 3D = 0\). Thus, \(D = 8\). In other words, to bring the population down to 6 thousand, we must kill skunks at the rate of 8 thousand a year.
14. Find a power series for each of the following functions.

(a) \( \frac{x}{1-2x} \).

Recall that for \( |r| < 1 \), \( \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \). Letting \( r = 2x \) and multiplying both sides by \( x \) gives

\[
\sum_{n=0}^{\infty} 2^n x^{n+1} = \frac{x}{1-2x}.
\]

This formula is valid for \( |x| < \frac{1}{2} \).

(b) \( \int_0^x e^{-t^2} \, dt \).

Since \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \),

\[
e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}.
\]

Integrating term-by-term

\[
\int_0^x e^{-t^2} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}.
\]

Since the series for \( e^x \) is valid for all \( x \), this formula is also valid for all \( x \).