1. (a) Find all \( r \) so that \( y = e^{rt} \) is a solution to the differential equation
\[ y'' + 3y' - 4y = 0. \]

(b) Find the general form of the solution to
\[ \frac{dy}{dx} = 1 + x + y + xy. \]

(c) Solve the initial value problem
\[ \frac{dy}{dx} = \frac{1}{y}, \quad y(0) = 1. \]

2. (a) Find the general solution of \( x^2 \frac{dy}{dx} = 1 + y - x^2 - x^2y. \)

(b) Solve the following initial value problem:
\[ \frac{dy}{dx} = y \ln(y)x, \quad y(0) = 4. \]

(c) Find the solution to the initial value problem
\[ \frac{dy}{dx} = (y - 1)(xy - x + y - 1), \quad y(0) = 0. \]

(d) Find the general solution to \( y^{1/3} \frac{dy}{dx} = yx \), simplifying your answer as much as possible.

3. Consider the differential equation \( y' = (y - 2)(1 - y) \).

(a) Sketch a direction field for this equation.

(b) List all equilibrium solutions.

(c) Find an explicit solution when \( y(0) = 3/2 \).

4. A tank holds 50 liters of water, with 100g of dye in it. To reduce the amount of dye, water containing 1g of dye per liter is stirred into the tank at a rate of 25 liters per minute, and the mixture is drained out at the same rate. How long does it take until the dye level drops below 51g?

5. Suppose that a certain species of lizard not native to California has escaped from a pet store. You are employed to monitor the population of this lizard in the Stanford foothills. Every year for 10 years you conduct a survey and estimate the population. Your obtain the following:
<table>
<thead>
<tr>
<th>year</th>
<th>Population (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.6</td>
</tr>
<tr>
<td>2</td>
<td>2.6</td>
</tr>
<tr>
<td>3</td>
<td>4.2</td>
</tr>
<tr>
<td>4</td>
<td>6.6</td>
</tr>
<tr>
<td>5</td>
<td>10.0</td>
</tr>
<tr>
<td>6</td>
<td>14.5</td>
</tr>
<tr>
<td>7</td>
<td>20.2</td>
</tr>
<tr>
<td>8</td>
<td>26.3</td>
</tr>
<tr>
<td>9</td>
<td>32.3</td>
</tr>
<tr>
<td>10</td>
<td>37.5</td>
</tr>
</tbody>
</table>

Assuming that the lizard population is governed by a logistic equation, estimate the level at which the population will stabilize.

6. (a) Find the general solution to $y'' - 3y' + 3y = 0$
(b) Find the general solution to $y'' - 2y' + 10y = 0$.
(c) Find the general solution to $y'' - y' + y = 0$.
(d) Find the solution of $y'' + y' - 2y = 0$ which satisfies $y(0) = 0$ and $y'(0) = 4$.
(e) Find the solution of $y'' + 6y' + 10y = 0$ which satisfies $y(0) = 4$ and $y'(0) = 0$.

7. Radioactive substances decay at a rate proportional to the remaining mass. A certain radioactive substance has a half-life of 100 years. Consider a sample with mass $M_0$.
(a) Find a formula for the mass that remains after $t$ years.
(b) How many years until only 5% of the original sample is left?

8. Suppose that a bank offers 5% continuously compounded interest, but charges a fee by removing money continuously from every account at the rate of $10 per year.
(a) Write the differential equation that governs accounts at this bank.
(b) Suppose that I deposit $1000 in this bank not knowing about the fee. How many more years will it take for my money to double than I expect?
9. (a) Find the solution to
\[
\frac{d^2y}{dt^2} - 3 \frac{dy}{dt} - 4y = 0
\]
with the initial data \( y(0) = 3 \) and \( y'(0) = 2 \).
(b) Consider the following system of differential equations:
\[
\begin{align*}
\frac{dx}{dt} &= -6x + 25y \\
\frac{dy}{dt} &= -x.
\end{align*}
\]
Convert it to a single second order equation in \( x \) and find the general solution to that equation.
(c) Find the general solution of \( \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 8x = 0 \).

10. Consider the following system of differential equations:
\[
\begin{align*}
\frac{dy}{dt} &= -y((x - 1)^2 + (y - 1)^2 - 1)(1 - y) \\
\frac{dx}{dt} &= -x((x - 1)^2 + (y - 1)^2 - 1)(1 - x)
\end{align*}
\]
By analyzing the nullclines etc. in the phase space, determine the long term behaviour of the system for various initial conditions.

11. The usual model for air resistance states that a body falling through air experiences an acceleration due to air resistance proportional to its velocity. Thus, an object falling to earth has acceleration \( a = g - kv \), where \( g \) is acceleration due to gravity and \( k \) is a positive constant. In this equation “down” is taken to be the positive direction. Take \( g = 10 \text{ m/sec}^2 \).
(a) For a skydiver with a closed parachute, \( k = \frac{1}{10} \). If Dave jumps out of an airplane, write an expression for his velocity as a function of time.
(b) After 10 seconds, Dave open’s his parachute. Now, \( k = 2 \). Write an expression for Dave’s velocity at time \( t > 10 \).
12. At a certain bank, the interest rate that you get, depends on the size of your balance. The higher the balance, the higher the interest rate. More precisely, if you have $A$ dollars in the bank, you get interest at the rate $\ln(A)$ percent a year. Assume the initial deposit is $M$. Write the differential equation satisfied by $A(t)$ and calculate how long it will take to reach $M^2$.

13. Consider the system of differential equations

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

Assume that $b$ and $c$ are nonzero.

(a) Show that $x$ and $y$ satisfy the same second-order differential equation.

(b) Assume that $ad - bc > 0$. Show that the system has a stable equilibrium if $a + d < 0$ and an unstable equilibrium if $a + d > 0$.

14. Some Stanford students meet their future husbands or wives while in college. For simplicity, assume that for a particular entering class, no one has yet met his or her future spouse when the class arrives. The rate at which students meet their future spouses is proportional to the square of how many have not yet met them. By the end of their senior year, one third of the students have met their future spouses. What fraction of students have met their future spouses by Valentine’s day of their Freshman year? (Valentine’s day is half-way through the year.)

15. In the Stanford foothills, field mice and chipmunks compete for the same resources. Let $M(t)$ be the number of field mice and $C(t)$ the number of chipmunks measured in tens of thousands. The following equations describe the rates of growth of the populations:

$$\frac{dM}{dt} = -4M^2 - MC + 4M$$
$$\frac{dC}{dt} = -\frac{1}{2}MC + C - C^2$$

Determine what happens in the long run if we start with 20,000 field mice and 10,000 chipmunks.