1. (a) Make the substitution \( u = t - 2 \). We get \( \int (u + 1)\sqrt{u} \, du = \int u^{3/2} + u^{1/2} \, du \). This equals \( 2u^{5/2}/5 + 3u^{3/2}/3 + C \). So, the answer is \( 2(t - 2)^{5/2}/5 + 3(t - 2)^{3/2}/3 + C \).

(b) The denominator factors as \( (x - 2)(x - 1) \). Using partial fractions we write the integrand as \( \frac{A}{x - 2} + \frac{B}{x - 1} \). To find \( A, B \), write \( x - 3 = A(x - 1) + B(x - 2) \). When \( x = 1 \) we get \( -2 = B(-1) \), so \( B = 2 \). When \( x = 2 \) we get \( A = -1 \). We are thus reduced to computing \( -\int \frac{dx}{x - 2} + 2 \int \frac{dx}{x - 1} \) which is \( -\ln |x - 2| + 2 \ln |x - 1| + C \).

(c) Make the substitution \( u = \sin^2 \theta + 1 \). Then \( \frac{1}{2} \, du = \sin \theta \cos \theta \, d\theta \).

The integral becomes \( \frac{1}{2} \int \frac{du}{u} \). The answer is \( \frac{1}{2} \ln(\sin^2 \theta + 1) + C \).

(d) First, observe that \( \ln(x^3) = 3 \ln(x) \). Thus, we must compute \( 9 \int (\ln(x)^2) \, dx \). We use integration by parts; \( f = (\ln(x)^2) \), \( g' = 1 \). So, \( f' = (\ln(x)) \, dx \) and \( g = x \). Thus we have \( 9[x(\ln(x)^2) - 2 \int \ln(x) \, dx] \). The antiderivative of \( \ln(x) \) may be computed by parts in a similar way; it is \( x \ln(x) - x \). Thus, the answer is \( 9x(\ln(x)^2) - 18x \ln x + 18x + C \).

2. (a) Use integration by parts with \( f = t \), \( g' = \sin(2t) \). Then \( f' = 1 \) and \( g = -\frac{1}{2} \cos(2t) \). The integral becomes \( -\frac{1}{2} \cos(2t)t^{\frac{7}{2}} + \frac{1}{2} \int_0^\pi \cos(2t) \, dt \). This equals \(-\frac{\pi}{2} + \frac{1}{4} \sin(2t)|_0^\pi = -\pi/2 \).

(b) Make the substitution \( u = \sin(3\theta) \). Then \( \frac{1}{3} \, du = \cos(3\theta) \, d\theta \) and the limits of integration are changed to \( 0 \) to \( 1/\sqrt{2} \). That is, we have \( \int_0^{\pi/2} u \, du \). This equals \( u^2/6|_0^{\pi/2} = 1/12 \).

(c) This is an improper integral because the integrand is undefined at \( \sqrt{2} \). Thus, we must compute \( \lim_{a \to \sqrt{2}} \int_0^a \frac{2x}{(x^2 - 2)^2} \, dx + \lim_{b \to \sqrt{2}} \int_b^\infty \frac{2x}{(x^2 - 2)^2} \, dx \).

Let us focus on the first integral. Using the substitution \( u = x^2 - 2 \), one checks that an antiderivative is \( -\frac{1}{2}(x^2 - 2)^{-1} \). Evaluating from \( 0 \) to \( a \), we get \( \lim_{a \to \sqrt{2}} -\frac{1}{2(a^2 - 2)^{2}} + 1/8 \). Since this limit is divergent, the original integral diverges.

3. Using the second form of the Fundamental Theorem of Calculus, we get \( 2x = \sqrt{1 + [f(x)]^2} \). Squaring both sides gives \( 4x^2 = 1 + |f(x)|^2 \) so \( f(x) = \sqrt{4x^2 - 1} \). We should now check that this \( f \) really satisfies the formula. The integral becomes \( \int_1^x 2t \, dt = t^2|_1^x = x^2 - 1 \) as needed.

4. Let \( G(y) = \int_{y_1}^y f(t) \, dt \). Then \( F(x) = -G(x^2 - 2x) \). Using the chain rule, this implies that \( \frac{dF}{dx} = -G'(x^2 - 2x)(2x - 2) \). Setting \( x = 1 \) gives \( \frac{dF}{dx}|_1 = -G'(-1)(0) = 0 \). It follows that \( x = 1 \) is a critical
point. (It is also worth observing that, by the Fundamental Theorem of Calculus, \( G' = f \).

5. We must compute \( \lim_{a \to \infty} \int_1^a \frac{\ln x}{x^r} \, dx \). First, assume that \( r = 1 \). In this case, we can make the substitution \( u = \ln x \) and are quickly reduced to \( \lim_{a \to \infty} (\ln a)^2/2 \) which clearly diverges. Now we may assume that \( r \neq 1 \). In this case we use integration by parts with \( f = \ln x \) and \( g' = x^{-r} \). So, \( f' = 1/x \) and \( g = x^{-r+1}/(-r+1) \). The integral then becomes \( \left[ \frac{\ln(x)x^{-r+1}}{(-r+1)} \right]_{1}^{a} - \frac{1}{(-r+1)} \int_1^a \frac{dx}{x^r} \). This equals \( \frac{\ln(a)a^{-r+1}}{(-r+1)} \) \( (a^{-r+1} - 1) \). Now, the limit as \( a \) goes to infinity of this expression diverges if \( r < 1 \) and converges if \( r > 1 \). All together, we have found that the integral converges only for those \( r \) which are strictly larger than 1.

6. \( \lim_{n \to \infty} \sum_{i=1}^{n} 21 \sin((-1 + \frac{3k}{n})^2)/n \).

7. (a) Let us first compute the antiderivative of the integrand. Making the substitution \( u = -3x^2 \) gives \( \frac{1}{18} \int u e^u \, du = \frac{1}{18} (ue^u - e^u) + C \). Now we compute the improper integral. We have \( \lim_{a \to \infty} \int_0^a x^3 e^{-3x^2} \, dx \). This equals \( \lim_{a \to \infty} \frac{1}{18} (-3x^2 e^{-3x^2} - e^{-3x^2}) \bigg|_0^a \). Plugging in \( a \) and taking the limit gives zero. Thus, the answer is \( 1/18 \).

(b) Making the substitution \( u = \cos(2\theta) \) reduces the integral to \( -\frac{1}{2} \int_1^{-1} u^2 \, du \). This equals \( -u^3/6 \bigg|^{-1}_1 = 0 \).

(c) The substitution \( u = 2 - x \) reduces the integral to \( -\int_2^1 u^{-1/2} \, du = \frac{2}{\sqrt{2}} - 2 \).

(d) The substitution \( u = \sin t \) changes the integral into \( \int_0^1 \frac{du}{\cos^2 u} \). Now we use partial fractions. We look for \( A, B \) so that \( 1 = A(u+2) + B(u-2) \). It is easy to see that \( A = 1/4 \) and \( B = -1/4 \). It follows that the definite integral equals \( \frac{1}{4} \int_0^1 \frac{du}{u-2} - \frac{1}{u+2} \). \( u \bigg|_{1/4} = \frac{1}{4} \ln \left| \frac{u-2}{u+2} \right| \). This equals \( \frac{1}{4} \ln(1/3) \).

(e) First, we compute the antiderivative using integration by parts twice. First, \( f = \cos(2x) \) and \( g' = e^{3x} \). So, \( f' = -2 \sin(2x) \) and \( g = e^{3x}/3 \). This gives \( \frac{1}{3} \cos(2x)e^{3x} + \frac{2}{3} \int \sin(2x)e^{3x} \, dx \). Integration by parts again gives \( \cos(2x)e^{3x}/3 + \frac{2}{3} \sin(2x)e^{3x} - \frac{1}{3} \int \cos(2x)e^{3x} \, dx \). This implies that \( \int \cos(2x)e^{3x} \, dx = \frac{9}{13} \left( \frac{1}{3} \cos(2x)e^{3x} + \frac{2}{3} \sin(2x)e^{3x} \right) + C \). Evaluating from \( \pi/4 \) to \( \pi/2 \) gives \( \frac{9}{13} \left( -\frac{1}{3} e^{3\pi/2} - \frac{2}{3} e^{3\pi/4} \right) \).

(f) Substituting \( u = 2x - 1 \) (so \( x = (u+1)/2 \)) gives \( \frac{1}{4} \int_{-3}^{-1} \frac{u+1}{u} \, du \). This equals \( \frac{1}{4} (u + \ln |u|) \bigg|_{-3}^{-1} = \frac{1}{4} (2 - \ln 3) \).
(g) This is an improper integral since the integrand is undefined at zero. We have \( \lim_{a \to 0^-} \int_a^2 x^{-2} \, dx + \lim_{b \to 0^+} \int_b^1 x^{-2} \, dx \). Focusing on the second limit, we have \( \lim_{b \to 0^+} -x^{-1}|_b \). This clearly diverges. Therefore, the improper integral diverges.

8. By definition, \( \int_0^1 x^3 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{1}{n} \right)^3 = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^3 \). Using the given formula, we get \( \lim_{n \to \infty} \frac{n^2(n+1)^2}{4n^3} = \lim_{n \to \infty} \frac{1 + 2/n + 1/n^2}{4} = \frac{1}{4} \).

9. It should look like the graph of \( (x^3 - 1)/3 \).

10. (a) We use partial fractions. We must find \( A \) and \( B \) which satisfy \( \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \); that is, \( x - 1 = A(x - 4) + Bx = (A + B)x - 4A \). Equating coefficients gives \( A = 1/4 \) and \( A + B = 1 \) so \( B = 3/4 \). It is now easy to see that the integral is \( \frac{1}{4} \ln |x| + \frac{3}{4} \ln |x - 4| + C \).

(b) Use integration by parts with \( f = \sin \theta \) and \( g' = \sin \theta \). Then \( f' = \cos \theta \) and \( g = -\cos \theta \). This gives \( -\sin \theta \cos \theta + \int \cos^2 \theta \, d\theta \). Using \( \cos^2 \theta = 1 - \sin^2 \theta \) and moving the integral of \( \sin^2 \theta \) to the left side, we get \( \frac{1}{2}(\theta - \sin \theta \cos \theta) \).

(c) We rewrite the problem as \( 9 \int y^{5/2} \, dy \). This equals \( \frac{18}{7} y^{7/2} + C \).

(d) Substituting \( u = \ln x \), \( du = x^{-1} \, dx \), we reduce to \( \int \cos u \, du \). The answer is \( \sin(\ln x) + C \).

(e) Substituting \( u = \cos(3\theta) \), \( -\frac{1}{3} \, du = \sin(3\theta) \, d\theta \), we reduce to \( -\frac{1}{3} \int \frac{\sin(3\theta)}{u} \, d\theta \). The answer is \( -\frac{1}{3} \ln |\cos(3\theta)| + C \).

(f) Making the substitution \( u = x + 1 \), we reduce to the expression \( \int \frac{(u-1)^2}{u^{1/2}} \, du \). This equals \( \int u^{3/2} - 2u^{1/2} + u^{-1/2} \, du \). Computing the antiderivative and then replacing \( u \) with \( x + 1 \), we get \( 2(x+1)^{5/2}/5 - 4(x+1)^{3/2}/3 + 2(x + 1)^{1/2}/2 + C \).

(g) Use integration by parts twice. First we get \( -x^2 \cos x + \int 2x \cos x \, dx \). Next, we reduce to \( -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \). Finally, the antiderivative is \( -x^2 \cos x + 2x \sin x + 2 \cos x + C \).

(h) Make the substitution \( u = 2 - 5x \). This reduces us to \( \frac{1}{5} \int \frac{du}{u^2} \). The answer is \( \frac{1}{5(2-5x)} + C \).

(i) Replace \( \cos^3 \theta \) by \( \cos \theta - \cos \theta \sin^2 \theta \) and split into two integrals. In each one, make the substitution \( u = \sin \theta \). The answer is \( (\sin^7 \theta)/7 - (\sin^9 \theta)/9 \).

(j) First make the substitution \( u = 2 + t^{1/2} \). Since \( u' = t^{-1/2} \), we are reduced to calculating the antiderivative of \( \ln u \) which can be done...
using integration by parts: $u \ln u - u + C$. The answer is obtained by substituting back in for $u$.

(k) We use partial fractions. We look for $A, B$ with \( \frac{1}{x-a}+\frac{B}{x-b} \) = \( \frac{A}{x-a}+\frac{B}{x-b} \). This leads to $1 = A(x-b)+B(x-a)$. Therefore, $A = 1/(a-b)$ and $B = -1/(a-b)$. It is now easy to see that the antiderivative is $\frac{1}{x-b}(\ln |x-a| - \ln |x-b|) + C$.

11. For all $x \geq 0$, $x + 1 \geq 1$. Thus, $\frac{e^{-x}}{x+1} \leq e^{-x}$. Moreover, $\int_0^\infty e^{-x}dx = \lim_{s \to \infty} 1 - e^{-s} = 1$. This shows that $\int_0^\infty \frac{e^{-x}}{x+1}dx$ is bounded by a convergent integral, and thus converges.

12. First, find the points of intersection of the two curves. For this, we look for positive values of $x$ which satisfy $-x^2 + 2x + 6 = 3x$ and negative values of $x$ which satisfy $-x^2 + 2x + 6 = -3x$. It is easy to see that $x = -1, 2$ work. We now must compute

$$\int_{-1}^{2} -x^2 + 2x + 6 - 3|x|dx = \int_{-1}^{0} -x^2 + 5x + 6dx + \int_{0}^{2} -x^2 - x + 6dx.$$  

This equals 21/2.

13. (a) $-4$ (b) $-2$ (c) 3 (d) 1 (first notice that $\int_0^3 g(x)dx = -1$.)

14. The integral represents the total amount of water used (in thousands of gallons) by the community between 10am and 10pm on April 2 1991.

15. By the Fundamental Theorem of Calculus, $g'(x) = p(x)(1 - x^2)$. Since, $p(x)$ is always positive, it follows that $g'(x)$ has the same sign as $1 - x^2$. Thus, $g(x)$ is increasing for $x$ in $(-1, 1)$, and $g(x)$ is decreasing for $x$ in $(-\infty, -1)$ and $(1, \infty)$.

16. We can rewrite the curve as $x = y/3$ for $0 \leq y \leq 6$ and $x = \sqrt{4y-20}$ for $6 < y \leq 9$. It is then easy to see that the volume is $\int_0^6 \pi (y/3)^2 dy + \int_6^9 \pi (4y-20) dy$. This equals $\frac{\pi}{3} y^3|_0^6 + 2\pi (y^2 - 10y)|_6^9 = 140\pi$ cubic inches.
17. Let us consider an upper hemisphere. Let \( s = 0 \) at the bottom (i.e. the center of the sphere); \( s = R \) at the top. We slice the object into pieces horizontally. By Pythagorean identity, the radius of a slice at height \( s_i \) is \( \sqrt{R^2 - s_i^2} \). It follows that the total volume is \( \int_0^R \pi (R^2 - s^2) \, ds = \pi (R^2 s - s^3/3)|_0^R = \frac{2}{3} \pi R^3 \).

18. The work to lift the X-wing, Jedi master and droid is \((10,000 + 75 + 50)10 = 101,250\) foot-lbs. Now the work to lift the mud. At \( h \) feet above the ground, there is \( 1000 - 50h \) pounds of mud. So, the work to move mud up to \( h + \Delta h \) is \((1000 - 50h) \Delta h \). So, total work for mud is \( \int_0^{10} (1000 - 50h) \, dh = 1000h - 25h^2|_0^{10} = 7500 \) foot pounds. The total work to lift everything is \( 108,750 \) ft-lbs.

19. Slicing the glass horizontally at various \( x_i \)'s into pieces of height \( \Delta x \), we can approximate the amount of beer as \( \sum \frac{x_i}{(x_i + 1)^2 \pi (1)^2 \Delta x} \). Taking the limit over more and more slices gives that the total amount of beer equals \( \frac{\pi}{2} \int_1^8 \frac{1}{x+1} \, dx \). To compute this, make the substitution \( u = x + 1 \). We get \( \frac{\pi}{2} \int_1^9 \frac{1}{u} \, du \). The answer is \( \frac{\pi}{2} (8 - \ln 9) \).

20. If the container was extended into a whole cone, it would bee 100 feet high. Let \( x \) measure feet up from the tip of the cone. Thus, \( x = 50 \) at the bottom of the container, and \( x = 100 \) at the top of the container. It is easy to see that the radius of the cone at level \( x \) is \( x/5 \). Thus, if we slice the container into pieces horizontally, the \( i \)th piece of water weighs approximately \( 62.4\pi (x_i/5)^2 \Delta x \). This water must be lifted \( 100 - x \) feet. The total work is

\[
\int_{50}^{100} 62.4\pi \left( \frac{x}{5} \right)^2 \left( 100 - x \right) \, dx.
\]

This equals \( \frac{62.4\pi}{25} (\frac{100}{3} x^3 - \frac{x^4}{4})|_{50}^{100} = \frac{62.4\pi}{3} 687,500 \) foot pounds.

21. The average temperature is \( \frac{1}{18-12} \int_{12}^{18} 80 + 10 \sin(\frac{\pi}{12}(t - 10)) \, dt = \frac{1}{6} \int_{12}^{18} 80 \, dt + \frac{10}{6} \int_{12}^{18} \sin(\frac{\pi}{12}(t - 10)) \, dt \). The first term is just 80. In the second integral, make the substitution \( w = \frac{\pi}{12}(t - 10) \). We get

\[
80 + \frac{20}{\pi} \int_{\pi/6}^{2\pi/3} \sin w \, dw = 80 + \frac{20}{\pi} (- \cos w)|_{\pi/6}^{2\pi/3} = 80 + \frac{10}{\pi}(\sqrt{3} + 1).
\]
22. (a) $10,000e^{21/20}$ (b) After $t$ years, the balance is $10,000e^{t/20}$. So, the average balance is $\frac{1}{21} \int_0^{21} 10,000e^{t/20} \, dt = 10,000 \frac{20}{21}(e^{21/20} - 1)$.

23. The balance after $T$ years is $\int_0^T 100e^{(T-t)/10} \, dt = 100(-10)e^{(T-t)/10}\bigg|_0^T = -1000(1-e^{T/10})$. Thus, the balance will be $\$1000$ when $e^{T/10} = 2$. That is, after $T = 10 \ln(2)$ years.