SAMPLE FINAL QUESTIONS
(This is longer than the final will be...)
No notes, no books.
YOU MUST SHOW ALL WORK TO RECEIVE CREDIT
Good luck!

Name ________________________________

ID number __________________________

1. ___________ (/?? points)

2. ___________ (/?? points)

3. ___________ (/?? points)

4. ___________ (/?? points)

5. ___________ (/?? points)

6. ___________ (/?? points)

Bonus ___________ (/?? points)

Total ___________ (/?? points)

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Signature: ____________________________

Circle your TA’s name:

Joe Blitzstein (2 and 7)
Eric Katz (3 and 8)
Robert Sussland (4 and 9)
Ben Howard (5 and 10)
Alex Meadows (12)
Dana Rowland (13)

Circle your section meeting time:

11:00am 1:15am 7pm
1. Evaluate the following integrals, using any exact technique (DO NOT evaluate it with your calculator, and do not give an estimate):

\[ \int_{0}^{3} 2 \sqrt{36 - 4x^2} \, dx \]

\[ \int_{-5}^{5} 2x \sqrt{e^{|x|}} \, dx \]
2. Derive the following antiderivatives (do not quote from memory, and do not use the guess-and-check method):

\[ \int \ln x \, dx \]

\[ \int e^x \sin x \, dx \]
3. Find the volume generated when the area bounded by $y = 4 - x^2$ and the $x$-axis is rotated about the line $x = 5$. 
4. Professor Curmudgeon is teaching a class in calculus. Over the years, he notes that the number of hour per week \((h)\) that he has to spend in his teaching duties depends only on the number of students in his class \((s)\). In particular,

\[ h(s) = 40 + 3\sqrt{s} \]

In the winter quarter (ten weeks), the class begins with 200 students; but because of his viciously difficult exams, the number of students drops off at a constant rate of 15 students per week, so that at the end of the quarter there are only 50 students remaining.

How many total hours does Curmudgeon have to spend in his teaching duties that quarter?
5. (For this problem, all payment rates and interest compoundings are continuous.)

a) You borrow $1,000,000 from the bank to purchase a small house in Palo Alto. The loan is to be repayed over thirty years, with a constant interest rate of 10%.

What is your payment rate? Express your answer in dollars per year. (Note: On the real exam, a problem like this would be arranged in such a way as to be easily computable.)

b) Then, immediately after you sign the papers, the bank realizes that the interest rate they should have given you was 12%; but, since the papers are already signed, they are obliged to let you keep the loan at the previously agreed upon rate.

In terms of present value, how much did this error profit you? In other words, what is the difference in the values of the payment stream from part (a) when calculated with the two different interest rates?
6. A service station in Palo Alto gets an average of ten customers per month (30 days) needing work done on their Mercedes automobiles. But, the only mechanic there who knows how to work on such a car wants to take a five day vacation. He agrees with the owner that he will wait until a Mercedes comes in, fix it, and then begin his vacation.

Assuming that the p.d.f. for the wait time for Mercedes automobiles coming into this service station is exponential, what is the probability that the station will not lose any business as a result of his absence? In other words, what is the probability that no Mercedes will be brought in during that five day period?
Professor Curmudgeon is again teaching calculus. It is observed that there is a relationship between the number of students that attend his lectures \((s)\), and his mood \((m)\). Having more students in his lectures upsets him because he knows he will have more work to do, so he gets grumpy (his mood lowers); this makes the attendance in his lectures lower because the students don’t like him, which in turn makes him happier (his mood raises), thus making the students like him more and attracting more students to come to his lectures... etc.

In particular, the differential equations governing this relationship are

\[
\frac{ds}{dt} = ms - \frac{s^2}{200}
\]

\[
\frac{dm}{dt} = 100m - ms
\]

a) Draw a slope-field diagram representing the movements in his mood and the attendance in his lectures. Include lines where either \(m\) or \(s\) are constant, and guide arrows representing the general directions of movement in each resulting region.

b) Is there an equilibrium in this system? If so, what are the values of \(s\) and \(m\) at this equilibrium?
8. Find the unique solution to the following initial value problem:

\[ \frac{dy}{dx} = 1 + x + y + xy \quad \text{with } y(1) = -2 \]
9. Suppose that the population of trout in a particular lake is governed by the following differential equation:

\[
\frac{dP}{dt} = 6P - P^2
\]

where \( P \) represents the population in thousands, and time \( t \) is measured in years. At present, the population is at the unique stable equilibrium value.

A few families move in to some nearby cabins, and they love to eat trout. In fact, they eat five thousand trout per year.

a) What will be the new equilibrium value for the population of trout in the lake?

b) What is the maximum number of trout per year that could be caught without the population dying out?
10. What is the minimum number of terms you must add in the series below in order to know for certain that your partial sum is within .001 of the actual sum \( S \)? Will that partial sum be greater than or less than the actual sum \( S \)?

\[
S = \sum_{i=1}^{\infty} (-1)^{n-1} \frac{1}{n^3} = \frac{1}{1} - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125} - \ldots
\]
11. Use the picture that motivates the Integral Test (shown in class) to find both lower and upper bounds for
the sum of the following series:

\[ \sum_{i=1}^{\infty} \frac{1}{i^2} \]
12. Which of the following converge? Which of the following converge absolutely? (Don’t forget, you must show all of your reasoning!)

\[ \sum_{1}^{\infty} \frac{1}{n \ln n} \]

\[ \sum_{1}^{\infty} \frac{(-1)^{n-1}(n + 1)}{n^2 - 5} \]

\[ \sum_{1}^{\infty} \frac{(-1)^{n}2^n}{n!} \]
13. Find the radii of convergence and intervals of convergence for the following power series.

\[ \sum_{n=1}^{\infty} \frac{(-3)^n x^n}{n^2 e^n} \]

\[ \sum_{n=0}^{\infty} \frac{(x-5)^{2n}}{3n + 1} \]

\[ \sum_{n=0}^{\infty} \frac{n! x^n}{(2n)!} \]
14.

a) Find a power series representation of the antiderivative of the function below. (Recall, this function does not have a “closed form” antiderivative.)

\[ f(x) = e^{-x^2/2} \]

b) Use your result from part (a) to find a series that converges to the probability that a random variable \( x \) (with a normal p.d.f. of mean 0 and standard deviation 1) will be between 0 and 1.
15. Find the Taylor polynomials of degree 3 (centered at \( x = 0 \)) of the following functions:

\[
f(x) = \ln(x + 3)
\]

\[
g(x) = (\sin x)e^x
\]

\[
h(x) = e^{x^2}
\]
**Bonus Question:** Prove that any polynomial is equal to its associated Taylor series centered at 0.