EXAM 1 SOLUTIONS
You have 2 hours.
No notes, no books.
YOU MUST SHOW ALL WORK TO RECEIVE CREDIT
Good luck!

Name ________________________________

1. ___________ (/20 points)

2. ___________ (/20 points)

3. ___________ (/15 points)

4. ___________ (/15 points)

5. ___________ (/15 points)

6. ___________ (/15 points)

Bonus ___________ (/10 points)

Total ___________ (/100 points)
1. Evaluate the following antiderivatives:

a)
\[ \int \frac{\cos(\ln x)}{x} \, dx \]

**Solution:**
We make the substitution \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \). So, we have

\[ \int \frac{\cos(\ln x)}{x} \, dx = \int \cos(u) \, du = \sin(u) + c = \sin(\ln x) + c \]

b)
\[ \int t^2 e^{2t} \, dt \]

**Solution:**
We integrate by parts, with \( f = t^2 \), and \( g' = e^{2t} \). Then we have \( f' = 2t \), and \( g = \frac{1}{2} e^{2t} \). So our antiderivative becomes

\[ t^2 \left( \frac{1}{2} e^{2t} \right) - \int 2t \left( \frac{1}{2} e^{2t} \right) \, dt = \frac{1}{2} t^2 e^{2t} - \int t e^{2t} \, dt \]

Again, we integrate by parts, this time with \( f = t \) and \( g' = e^{2t} \). So \( f' = 1 \) and \( g = \frac{1}{2} e^{2t} \). Our antiderivative becomes

\[ \frac{1}{2} t^2 e^{2t} - \int t e^{2t} \, dt = \frac{1}{2} t^2 e^{2t} - \left[ t \left( \frac{1}{2} e^{2t} \right) - \int \frac{1}{2} e^{2t} \, dt \right] \]

\[ = \frac{1}{2} t^2 e^{2t} - \frac{1}{2} t e^{2t} + \frac{1}{2} \int e^{2t} \, dt \]

\[ = \frac{1}{2} t^2 e^{2t} - \frac{1}{2} t e^{2t} + \frac{1}{4} e^{2t} + c \]

\[ = \frac{e^{2t}}{4} (2t^2 - 2t + 1) + c \]
2. Find the following integrals:

\[ \int_{1}^{e} x^2 \ln x \, dx \]

**Solution:**
First we find the antiderivative, by integrating by parts. We choose \( f = \ln x \), and \( g' = x^2 \), giving us \( f' = \frac{1}{x} \), and \( g = \frac{1}{3} x^3 \). Our antiderivative then becomes

\[
\int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln x - \int \left( \frac{1}{x} \right) \frac{1}{3} x^3 \, dx
\]
\[
= \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^2 \, dx
\]
\[
= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3
\]
So, our integral is then

\[
\int_{1}^{e} x^2 \ln x \, dx = \left( \frac{1}{3} e^3 \ln e - \frac{1}{9} e^3 \right) - \left( \frac{1}{3} \ln 1 - \frac{1}{9} \right)
\]
\[
= \frac{1}{3} e^3 - \frac{1}{9} e^3 + \frac{1}{9} = \frac{2}{9} e^3 + \frac{1}{9}
\]

b)

\[ \int_{\pi}^{2\pi} (\sin t) \left( e^{\cos t} \right) \, dt \]

**Solution:**
Seeing the composition in the integrand, we substitute the inside function, \( u = \cos t \). Then \( du = -\sin t \, dt \), so that \(-du = \sin t \, dt \). The antiderivative then becomes

\[
\int e^u \, (-du) = - \int e^u \, du = -e^u = -e^{\cos t}
\]
So, the integral is then

\[
\int_{\pi}^{2\pi} (\sin t) \left( e^{\cos t} \right) \, dt = \left( -e^{\cos 2\pi} \right) - \left( -e^{\cos \pi} \right)
\]
\[
= -e^1 + e^{-1}
\]
\[
= -e + \frac{1}{e}
\]
3. Find the antiderivative.

\[ \int \frac{2x^2 + x + 1}{(x^2 + 1)(x - 1)} \, dx \]

**Solution:**
We use the method of partial fractions to try to break this up into two separate integrals.

\[
\frac{2x^2 + x + 1}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1}
\]

\[ 2x^2 + x + 1 = (Ax + B)(x - 1) + (C)(x^2 + 1) \]

\[ = (A + C)x^2 + (-A + B)x + (-B + C) \]

Plugging in \( x = 1 \) into the second equation above tells us immediately that \( C = 2 \); then, equating coefficients in the last equation gives us \( A + C = 2 \implies A = 0 \), and then \(-A + B = 1 \implies B = 1 \). So we have

\[
\frac{2x^2 + x + 1}{(x^2 + 1)(x - 1)} = \frac{1}{x^2 + 1} + \frac{2}{x - 1}
\]

So the antiderivative becomes

\[
\int \frac{2x^2 + x + 1}{(x^2 + 1)(x - 1)} \, dx = \int \frac{1}{x^2 + 1} + \frac{2}{x - 1} \, dx
\]

\[ = \int \frac{1}{x^2 + 1} \, dx + \int \frac{2}{x - 1} \, dx \]

\[ = \arctan(x) + 2 \ln |x - 1| + c \]
4. Use the error bound formula for Simpson’s Rule given here

\[ |E_{S_n}| \leq \frac{K_4(b-a)^5}{180n^4} \quad \text{(where} \quad K_4 \geq |f^{(4)}(x)| \quad \text{on} \quad [a,b]) \]

to show that Simpson’s Rule (using any value of \( n \)) provides an exact answer when estimating integrals of polynomials of degree three or lower – in other words, functions of the form

\[ f(x) = ax^3 + bx^2 + cx + d \]

(Recall that \( f^{[4]} \) means the fourth derivative of the function \( f \).)

**Solution:**
First, notice that for any function \( f \) of the given form, we have that \( f^{[4]} = 0 \) on the whole interval from \( a \) to \( b \).

So, since \( K_4 \) can be any number that is greater than or equal to \( |f^{[4]}| \) on the whole interval, we can choose \( K_4 = 0 \).

Plugging this into the given formula for the error bound, we get

\[ |E_{S_n}| = \frac{0(b-a)^5}{180n^4} = 0 \quad \text{(where} \quad K_4 \geq |f^{[4]}(x)| \quad \text{on} \quad [a,b]) \]

We conclude that the error \( E_{S_n} \) must be zero, which means that the estimate \( S_n \) is exact.
5. Use the Evaluation Theorem to prove FTC-I. (You may assume that the function $f$ has an antiderivative.) For your reference, here are the statements of those theorems:

**Evaluation Theorem:**
If $f$ is continuous on $[a, b]$, and $F$ is any antiderivative of $f$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

**FTC-I:** If $f$ is continuous on $[a, b]$, and

$$g(x) = \int_a^x f(t) \, dt$$

then $g'(x) = f(x)$.

**Solution:**
We need only compute the derivative of $g$; as we did in class, we use that Evaluation Theorem as suggested to evaluate $g$ before taking the derivative.

We assume that $f$ has an antiderivative; call it $F$. Then, by the Evaluation Theorem, we have

$$g'(x) = \left( \int_a^x f(t) \, dt \right)' = (F(x) - F(a))'$$

We compute these resulting derivatives directly with the chain rule.

$$= F'(x)(x)' - F'(a)(a)' = F'(x) = f(x)$$
6. Determine if the following integral converges or diverges:

\[ \int_1^\infty |\sin(e^{x^2+3\ln x})| e^{-x} \, dx \]

**Solution:**
We use the Comparison Theorem. Since the dominant part of the integrand is \( e^{-x} \), and since

\[ \int_1^\infty e^{-x} \, dx \]

converges, we suspect that the given integral probably converges also. In fact, we also notice that

\[ |\sin(e^{x^2+3\ln x})| e^{-x} \leq e^{-x} \]

Both of the above functions are positive, so we can apply the Comparison Theorem. We conclude that the given integral converges.
**Bonus Question:** Prove the following.

The Limit Comparison Theorem

If $f$ and $g$ are continuous, positive functions for all values of $x$, and

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = k \quad \text{(with } 0 < k < \infty) \]

and if $\int_a^\infty g(x) \, dx$ converges, then so does $\int_a^\infty f(x) \, dx$.

**Solution:**

(Note: This was also done in class!) The definition of the limit tells us that there exists some $N$ such that

\[(k - 1) < \frac{f(x)}{g(x)} < (k + 1) \quad \text{whenever } x > N.\]

So, for those values of $x$, we have

\[\frac{f(x)}{g(x)} < (k + 1) \implies f(x) < (k + 1)g(x)\]

We now break the integral in question into two pieces:

\[\int_a^\infty f(x) \, dx = \int_a^N f(x) \, dx + \int_N^\infty f(x) \, dx\]

The first integral is of a continuous function on a closed, bounded interval, so we know that is finite. The convergence of the second integral is concluded by the following, which we can do because of the inequality determined above:

\[\int_N^\infty f(x) \, dx < \int_N^\infty (k + 1)g(x) \, dx = (k + 1) \int_N^\infty g(x) \, dx\]

(the last integral in the equation above is given to converge; therefore, by the Comparison Theorem, the integral on the left converges.)

We conclude, as desired, that the integral of $f$ converges.