SECOND SAMPLE FINAL EXAM
MATH 41

Fall, 1999
Three Hours

NAME:

Section Number:

I agree to abide by the Honor Code.
Signature:

Instructions: Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. For each of the following, compute $\frac{dy}{dx}$.

(a) $y = \frac{2e^{3x}}{\ln(x^2)}$

(b) $y = x\cos^2(x^3 - 7x)$
2. In Math41Land, there is a large river, called LargeRiver, which runs in a straight east-west course. Town $A$ is located 2 miles north of LargeRiver, and town $B$ is located 3 miles north and 4 miles west of town $A$. The citizens of $A$ and $B$ have agreed to build a port on LargeRiver at a location which minimizes the sum of the distances from $A$ and $B$ to the port. Where should the port be located?
3. (a) Suppose that $\log_{h}(\frac{x}{x^2}) + x + 2 = 0$. Write $h$ as a function of $x$. Here $x > 0$.

(b) Consider the graph of $y = h(x)$ (from part (a)). Suppose that this curve is reflected about the line $y = 1$ and shifted three units to the right. Write an expression for the function whose graph is this new curve.
4. Compute the following limits.

(a) \( \lim_{x \to 1} \frac{x^2 - 2x}{1 - \ln x} \)

(b) \( \lim_{x \to 1} \frac{\sqrt[3]{2 - x^3} - x}{1 - \sqrt[3]{x^3}} \)
(c) \( \lim_{x \to \infty} \frac{\sqrt{2 + 5x^2}}{x} \)
5. Prove by induction that for any positive integer $n$, and any $n$
differentiable functions $f_1, f_2, \ldots, f_n$,

$$(f_1 \cdot f_2 \cdot \cdots \cdot f_n)' = (f_1' \cdot f_2 \cdot \cdots \cdot f_n) + (f_1 \cdot f_2' \cdot f_3 \cdot \cdots \cdot f_n) + \cdots + (f_1 \cdot f_2 \cdot \cdots \cdot f_n').$$
6. A water tank has the shape of a cone (point downwards) with radius 10 feet and height 10 feet. It is partially filled with water. A device also in the shape of a cone, radius 1 foot and height 2 feet, is lowered (point first) into the water at 1 foot per minute. How fast is the water level in the tank rising at the instant when the instrument is submerged half way (i.e. 1 foot), assuming that the water level in the tank is 8 feet at that time? [Remarks. (1) Water will not spill over the top of the tank. (2) In this problem, it is especially important to explain what all of your variables represent.]
7. Suppose that \( f'(x) = x^2e^{-x} + 1 \).

(a) What is the minimum value of \( f'(x) \) for \( x \) in \([-1, 2]\)?

(b) Suppose that \( f(-1) = 0 \). Use proof by contradiction to show that \( f(2) \geq 3 \). (Hint: use part (a) and the mean value theorem.)
8. Find a function $f$ so that $f'(x) = x^3$ and the line $8x + y = 0$ is tangent to the graph of $f$. 
9. Let

\[ f(x) = \begin{cases} 
\frac{2}{x} + a^2x & 0 < x < 1 \\
ax + 2a & x \geq 1.
\end{cases} \]

Determine for which value(s) of \( a \), \( f'(x) \) is defined and continuous at all \( x > 0 \).
10. (a) Use the definition of derivative to prove that \( \frac{d}{dx}(x^3) = 3x^2 \).

(b) Using the fact that \( \frac{d}{dx}(e^x) = e^x \), prove that \( \frac{d}{dx}(a^x) = a^x \ln a \) for any \( a > 0 \).
11. What is the maximum possible area for an isosceles triangle with perimeter $P$? (You must prove your answer!)
12. (a) State the precise definition of \( \lim_{x \to a} f(x) = L \).

(b) Using the definition, prove that \( \lim_{x \to 2} (x - 2)^4 = 0 \).
13. A red ball is dropped out a window 100 feet above the ground. One second later, a yellow ball is thrown upward from a window 20 feet above the ground. If the two balls hit the ground at the same instant, how fast was the yellow ball thrown? (Note that gravity provides a downward acceleration of 32feet/sec².)
14. Below is the graph of \( \frac{df}{dx} \) (NOT \( f(x) \)))! Note that the curve crosses the \( x \)-axis at \(-\sqrt{3}, 0\) and \( \sqrt{3} \). For each of the following questions, circle all that apply.

(a) \( f(x) \) has a local minimum at:
\(-\sqrt{3} \quad -1 \quad 0 \quad 1 \quad \sqrt{3}\)

(b) \( f(x) \) is concave down on:
\((-2, -\sqrt{3}) \quad (-\sqrt{3}, -1) \quad (-1, 0) \quad (0, 1) \quad (1, \sqrt{3}) \quad (\sqrt{3}, 2)\)

(c) \( f'(x) \) is concave up on:
\((-2, -\sqrt{3}) \quad (-\sqrt{3}, -1) \quad (-1, 0) \quad (0, 1) \quad (1, \sqrt{3}) \quad (\sqrt{3}, 2)\)

(d) at \( x = 0 \), \( f''(x) \) has a critical number inflection point discontinuity local maximum local minimum