SECOND SAMPLE MIDTERM #1
MATH 41

FALL, 1999

NAME:

Section Number:

I agree to abide by the Honor Code.
Signature:

**Instructions:** Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. Consider the function \( f(x) = \sqrt{x^2 + 1} \).
   (a) Using the definition of derivative, determine whether \( f'(0) \) exists, and if it does, find its value.

(b) Using the tangent line approximation, estimate \( \sqrt{1.1} \).
2. (a) Is \( \frac{\ln 2}{\ln 8} \) rational or irrational? How do you know?

(b) Prove that \( \log_2 6 \) is an irrational number.
3. (a) Does \( f(x) = \sqrt{x^2 + 1} - \sqrt{x^2 + 5x} \) have a limit as \( x \to \infty \)? If so, what is it?

(b) Find \( \lim_{x \to 0} \left( e^{-1/x^2} \cos(1/x^2) \right) \).
(c) Let $f$ be a function and $a$ a number in the domain of $f$. Consider the following argument:

$$
\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} (x - a)
$$

$$
= \left( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left( \lim_{x \to a} (x - a) \right).
$$

Since $\lim_{x \to a} (x - a) = 0$, it follows that $\lim_{x \to a} (f(x) - f(a)) = 0$. In other words, $f$ is continuous at $a$. This seems to show that every function is continuous at every point of its domain. But this is nonsense. What’s wrong?
4. (a) Suppose that \( \ln(s - 1) - 6 = 2 \ln(\frac{t}{e^t}) \). Write \( s \) as a function of \( t \). Here \( s > 1, t > 0 \).

(b) Using the definition of derivative, calculate \( s'(2) \).

(c) Write the equation for the tangent line to the graph of \( s(t) \) when \( t = 2 \).
5. Suppose that a cone is being filled with water coming through a hose at a constant rate. If the depth of the water, measured from the tip, is \( h(t) \), what can you say about \( h'(t) \) and \( h''(t) \)? Sketch a possible graph of \( h'(t) \).
6. Suppose that $f$ is a function that satisfies the equation
\[ f(x + y) = f(x) + f(y) + 3x^2y + 3xy^2 \]
for all real numbers $x$ and $y$. Also, assume that
\[ \lim_{x \to 0} \frac{f(x)}{x} = 1. \]

(a) Find $f(0)$.

(b) Find $f'(0)$.

(c) Find $f'(x)$. 
7. (a) Sketch the graph of a function $f(x)$ which satisfies $f'(x) > 0$ for $x < -2$ or $x > 5$, $f'(x) < 0$ for $-2 < x < 2$, and $f'(x) = 0$ for $2 \leq x \leq 5$. Also, the graph of $f$ lies above its tangent lines for $x < -3$ and below its tangent lines for $-3 < x < 1$ and $x > 7$. 
(b) Below is a graph of $g''(t)$. Sketch graphs of $g'(t)$ and $g(t)$ with $g(1) = g'(1) = 0.$
8. (a) State the definition of \( \lim_{x \to \infty} f(x) = L \).

(b) Using the definition, prove that \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \).