FIRST SAMPLE MIDTERM #1  
MATH 41  
FALL, 1999

NAME:  

SOLUTIONS

Section Number:

I agree to abide by the Honor Code.  
Signature:

SOLUTIONS

Instructions: Show all work. Unless a numerical approximation is specifically requested, an EXACT solution is required.

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1. Let \( g(x) = \frac{2}{x} \). In this problem we consider secant and tangent lines to the graph of \( g(x) \).

(a) Write an equation for the slope of the secant line between \((2, 1)\) and \((x, \frac{2}{x})\).

\[
\text{The slope of the line through (2,1) and } (x, \frac{2}{x}) \text{ is}
\]
\[
\frac{\frac{2}{x} - 1}{x - 2} = \frac{2 - x}{x(x - 2)}.
\]
Canceling gives
\[
-\frac{1}{x}.
\]

(b) Find the slope of the secant line between \((2, 1)\) and \((3, \frac{3}{2})\).

Plugging in \( x = 3 \) in the answer to part (a) gives
\[
-\frac{1}{3}.
\]

(c) Find the slope of the tangent line at \((2, 1)\).

\[
\text{The slope of the tangent line is the limit as } x \rightarrow 2 \text{ of the slope of the secant line. That is, the slope of the tangent line}
\]
\[
= \lim_{x \rightarrow 2} \frac{-1}{x} = -\frac{1}{2}.
\]

(d) Find the function whose graph is the tangent line at \((2, 1)\).

Since its graph is a line, the function must have the form \( f(x) = mx + b \). From part (c), we know that \( m = -\frac{1}{2} \). Also, the line goes through \((2, 1)\) so \( f(2) = 1 \). Thus,
\[
1 = f(2) = -1 + b.
\]
and so \( b = 2 \). Thus,
\[
f(x) = -\frac{1}{2}x + 2.
\]
2. Calculate each of the following limits. If the limit does not exist, say so and then calculate the left-hand and right-hand limits.

NOTE: Show your work and explain what you are doing!

(a) \( \lim_{y \to 3} \frac{1}{y} - \frac{1}{3} \)

Since we are interested in values of \( y \) near 3, we may assume that \( y \neq 0 \) and so multiply by \( \frac{3y}{3y} \). We get

\[
\lim_{y \to 3} \frac{3 - y}{(y - 3)3y}.
\]

Since we are not interested in \( y = 3 \), we may cancel the \((y - 3)\)'s and get

\[
\lim_{y \to 3} \frac{1}{3y} = \frac{1}{9}.
\]

(b) \( \lim_{x \to 0} \frac{x}{x - 2|x|} \)

Let us consider the left- and right-hand limits. (Note that we never consider \( x = 0 \).)

\[
\lim_{x \to 0^-} \frac{x}{x - 2|x|} = \lim_{x \to 0^-} \frac{x}{x + 2x} = \lim_{x \to 0^-} \frac{1}{3} = \frac{1}{3}.
\]

And

\[
\lim_{x \to 0^+} \frac{x}{x - 2|x|} = \lim_{x \to 0^+} \frac{x}{x - 2x} = \lim_{x \to 0^-} -1 = -1.
\]

Since the left- and right-hand limits are not equal, \( \lim_{x \to 0} \frac{x}{x - 2|x|} \) does not exist.
(c) \( \lim_{\theta \to \pi/4} \sin(\theta) + \sqrt{1 + \cos^2(\theta)} \)

First note that \( \lim_{\theta \to \pi/4} \sin(\theta) = \lim_{\theta \to \pi/4} \cos \theta \, 1/\sqrt{2} \). It then follows from the limit laws that our limit equals

\[
\lim_{\theta \to \pi/4} \sin(\theta) + \sqrt{1 + \left( \lim_{\theta \to \pi/4} \cos(\theta) \right)^2} = \frac{1}{\sqrt{2}} + \sqrt{1 + \frac{1}{2}}.
\]

This simplifies to \( \frac{1 + \sqrt{3}}{\sqrt{2}} \).

(d) \( \lim_{r \to \infty} \frac{3r + 1}{2r^2 - 5} \)

Since the limit of the numerator and the denominator do not exist, we can not use the limit laws directly. So, we first divide top and bottom by \( r \). We get

\[
\lim_{r \to \infty} \frac{3 + 1/r}{2r - 5/r}.
\]

Obviously, the limit of the numerator is now 3. However, the denominator blows up as \( r \to \infty \). It follows that the limit of the ratio is 0.
(e) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

First, rationalize the numerator by multiplying by $\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$. This gives

$$\lim_{x \to 0} \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})}.$$ 

Since $x \neq 0$, we can cancel the $x$. We get

$$\lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2}{2} = 1.$$
3. (a) Express the area, $A$, of an isosceles right triangle as a function of its perimeter, $p$.

Let us first introduce some notation. Let $h$ be the length of the hypotenuse and $l$ the length of each of the other sides. Obviously, $A = \frac{1}{2}l^2$. It follows that if we can express $l$ as a function of $p$, we can express $A$ as a function of $p$. Now, by definition, $p = h + 2l$. Furthermore, by the Pythagorean identity, $h^2 = 2l^2$. Thus, $p = \sqrt{2}l + 2l$. In other words, $l = p/(2 + \sqrt{2})$. Plugging this into our expression for $A$ gives $A = \frac{1}{2(2 + \sqrt{2})}p^2$. Thus we have

$$A(p) = \frac{1}{12 + 8\sqrt{2}}p^2$$

(b) Using the definition of the derivative, find $\frac{dA}{dp}$.

By definition,

$$\frac{dA}{dp} = \lim_{\Delta \to 0} \frac{A(p + \Delta) - A(p)}{\Delta}.$$ 

Now, by part(a), $A(p + \Delta) - A(p) = \frac{1}{12 + 8\sqrt{2}}((p + \Delta)^2 - p^2) = \frac{1}{12 + 8\sqrt{2}}(2p\Delta + \Delta^2)$. Thus,

$$\frac{dA}{dp} = \frac{1}{12 + 8\sqrt{2}} \lim_{\Delta \to 0} \frac{2p\Delta + \Delta^2}{\Delta} = \frac{1}{12 + 8\sqrt{2}} \left(2p + \lim_{\Delta \to 0} \Delta\right).$$

The last limit is obviously zero so we are left with

$$\frac{dA}{dp} = \frac{1}{6 + 4\sqrt{2}}p.$$
4. Let $f$ be a function whose domain is all positive integers and which satisfies

$$f(xy) = f(x) + f(y)$$

for all positive integers $x, y$.

(a) Prove that $f(1) = 0$.

Let $x = y = 1$. Then, by the given relation,

$$f(1) = f(1) + f(1).$$

Subtracting $f(1)$ from both sides gives

$$f(1) = 0$$
as claimed.

(b) Suppose further that for all prime numbers $p$, $f(p) = \ln(p)$. Prove that $f(x) = \ln(x)$ for all positive integers $x$.

First, observe that, by part (a), the statement is true for $x = 1$. For $x > 1$, we will use induction, so assume that $f(y) = \ln(y)$ for all positive integers $y < x$; using this we will establish that $f(x) = \ln(x)$.

Now there are two possibilities: either $x$ is a prime number or it is not. If $x$ is prime then $f(x) = \ln(x)$ by assumption so there is nothing to prove. If $x$ is not prime then, by the definition of prime number, it is the product of two smaller integers. More precisely, we may write $x = yz$ where $y, z$ are positive integers less than $x$. Because they are smaller than $x$, we know that $f(y) = \ln(y)$ and $f(z) = \ln(z)$. Thus,

$$f(x) = f(yz) = f(y) + f(z) = \ln(y) + \ln(z).$$

Finally, by a property of logarithms, $\ln(y) + \ln(z) = \ln(yz) = \ln(x)$. We have now established that $f(x) = \ln(x)$ for any positive integer $x$. 

5. In each part of this problem there is a statement about a function $h(x)$. On the next page there are four possible graphs of $h(x)$. In each part, list the roman numerals corresponding to ALL graphs which make the statement true; there may be more than one. If none of the graphs make the statement true, write “NONE”.

(a) $h(0) = -2$  

(b) $\lim_{x \to -1} h(x) = 1$ 

(c) $\lim_{x \to 1} h(x)$ does not exist 

(d) $-2, -1, 0, 1, 2$ are ALL in the domain of $h(x)$ 

(e) $\lim_{x \to 0^-} h(x) > 0$ 

(f) $\lim_{x \to 2} h(x)$ does not exist
6. (a) Let $\epsilon > 0$. Find $\delta > 0$ such that
\[ |x^3 - 8| < \epsilon \text{ whenever } 0 < |x - 2| < \delta. \]

Observe that $|x^3 - 8| = |x - 2||x^2 + 2x + 4|$. Thus, $|x^3 - 8|$ will be smaller than $\epsilon$ as long as $|x^2 + 2x + 4|$ is less than some constant $C$ and $|x - 2| < \epsilon / C$. But, as long as $0 < |x - 2| < 1$, then $1 < x < 3$ so $|x^2 + 2x + 4| < 9 + 6 + 4 = 19$. Hence, $C = 19$ will work and we can take $\delta = \min\{1, \epsilon / 19\}$.

(b) Use your answer from part (a) to prove that
\[ \lim_{x \to 2} x^3 = 8. \]

Given $\epsilon > 0$, let $\delta = \min\{1, \epsilon / 19\}$, and assume that $0 < |x - 2| < \delta$. We must show that this implies $|x^3 - 8| < \epsilon$. We will use the fact that $|x^3 - 8| = |x - 2||x^2 + 2x + 4|$. Of course, $|x - 2| < \delta \leq \epsilon / 19$. Also, $0 < |x - 2| < 1$ implies $1 < x < 3$ so $|x^2 + 2x + 4| < 9 + 6 + 4 = 19$. Thus, as long as $0 < |x - 2| < \delta$, $|x^3 - 8| < (\epsilon / 19)19 = \epsilon$. 
7. (a) Sketch the graph of a function $g(x)$, $0 \leq x \leq 2$, which satisfies $g(0) = 10$, $g(1) = 0$ and $g'(x)$ is constant.

(b) Sketch the graph of a function $f(x)$, $0 \leq x \leq 2$, which satisfies $f(0) = 0$, $f(x) \geq 0$ for all $x$, $f'(x) > 0$ for $0 < x < 1$, $f'(x) < 0$ for $1 < x < 2$, and $f''(x)$ is constant.

(c) Which of the functions $f$ or $g$ above is more plausible as the velocity of a ball thrown upward from the ground? Explain.
A ball thrown upward from the ground begins with positive (upward) velocity, gradually slows down, and finally begins to descend, gaining larger and larger negative velocity. This is consistent with the function $g$ but not with the function $f$. Indeed, a ball thrown upward does not begin with zero velocity and then have its velocity increase. $f$ would be more plausible as the position of the ball.