Extra Problem Set #1
1. Below is the graph of a function $h(x)$. In each part of this problem there is a statement about the behavior of $h(x)$ at $x = a$. List ALL values of $a$ which make the statement true; there may be more than one. If there are no such values, write “NONE”.

VALUES FOR $a$:

(a) $h(x)$ has a removable discontinuity at $x = a$  
(b) $h(x)$ has a non-removable discontinuity at $x = a$  
(c) $h(x)$ is continuous but not differentiable at $x = a$  
(d) $h(x)$ is differentiable but not continuous at $x = a$
2. A certain company manufactures and sells widgets. If they produce up to five widgets, they can sell them for $10.00 each. For each additional widget they produce, they need to lower the price of all widgets by 20 cents in order to sell them. Write an expression for the company’s revenues (i.e. the total amount they get for their widgets) as a function of the number of widgets they make. You may assume that they produce at least five widgets and sell all of them.

3. Using the definition, prove that \( \lim_{x \to \infty} e^{-x/7} = 0. \)

4. Use the intermediate value theorem to show that \( x^3 - 3x^2 + 1 \) has three distinct roots.

5. Let \( f \) be a function which is differentiable at 0, and satisfies \( f(0) = 0 \) and

\[
\lim_{x \to \infty} \frac{x^2 f(\frac{1}{x})}{3x + 2} = 1.
\]

Find \( f'(0) \).

6. (a) Solve the following equation for \( x \): \( \ln(\ln(x^2)) + \ln(\frac{1}{2}e) = 2. \)

(b) By inspection, find two values for \( x \) so that \( x^2 = 2^x \).

7. Suppose that we are interested in a parametric curve but we do not have formulas for \( x(t) \) or \( y(t) \). However, we do know that when \( t = 1 \) the curve passes through the point \((3, 7)\), and that \( \frac{dx}{dt} = e^{t^2} \) and \( \frac{dy}{dt} = e^{-t^2} \). Use linear approximation to estimate which point the curve passes through when \( t = 1.5 \).

8. Find the following limits. Show your work and explain what you are doing.

(a) \( \lim_{x \to \infty} \frac{8x^3 + 1}{14x^2 + 2x - 6} \)

(b) \( \lim_{\theta \to \frac{\pi}{2}} \frac{\cos \theta}{\sin \theta} \)
(c) \[ \lim_{t \to 1} \frac{1/t - 1/(t - 1)}{4/(t^2 + 2t - 3)} \]

(d) \[ \lim_{n \to \infty} \frac{\cos(2\pi n)}{n} \]

9. Consider the following condition on a positive integer \( p \): whenever \( x \) and \( y \) are positive integers and \( p \) divides \( xy \), then \( p \) divides at least one of \( x \) and \( y \). Prove that any \( p \) satisfying this condition is a prime number.

10. Below is the graph of \( \frac{df}{dx} \) (NOT \( f(x) \)). Note that the curve crosses the \( x \)-axis at \(-\sqrt{3}, 0\) and \( \sqrt{3} \). For each of the following questions, circle all that apply.
   (a) \( f''(x) = 0 \) at:
       \(-\sqrt{3} \quad -1 \quad 0 \quad 1 \quad \sqrt{3} \)

   (b) \( f(x) \) is increasing on:
       \(-\infty, -\sqrt{3}\) \(-\sqrt{3}, -1\) \((-1, 0)\) \((0, 1)\) \((1, \sqrt{3})\) \((\sqrt{3}, \infty)\)

   (c) \( f(x) \) is concave down on:
       \(-\infty, -\sqrt{3}\) \(-\sqrt{3}, -1\) \((-1, 0)\) \((0, 1)\) \((1, \sqrt{3})\) \((\sqrt{3}, \infty)\)

   (d) \( f'(x) \) is this kind of function:
       exponential logarithmic odd even one-to-one
11. Consider the following argument: Let \( f(x) = \frac{x^2 + 1}{x^3 - x - 1} \). It is easy to check that \( f(0) = -1 \) and \( f(2) = 1 \). Therefore, by the intermediate value theorem, there is a number, \( a \), between 0 and 2 with \( f(a) = 0 \). In other words, \( a^2 + 1 = 0 \). Clearing the denominator, \( a^2 + 1 = 0 \), or, in other words, \( a^2 = -1 \). This seems to give a proof that \(-1\) has a real square-root. But this is nonsense. What’s wrong?

12. Simplify each of the following expressions.
(a) \( e^{2\ln(x)+5\ln(y)} \)
(b) \( \ln\left(\frac{1}{e^{y^2}}\right) \)
(c) \( e^{\frac{\ln\left(\ln x\right)}{\ln e}} \)

13. Use induction to prove that the sum of the integers less than \( 3n \), which are not divisible by 3, is \( 3n^2 \); i.e.

\[
1 + 2 + 4 + 5 + 7 + 8 + \cdots + (3n - 2) + (3n - 1) = 3n^2.
\]
14. Let \( f(x) = x^2 \) and \( g(x) = \sqrt{x + 1} \).

(a) Calculate \( f \circ g \) and find its domain.

(b) Find a function \( h(x) \) so that

\[
g \circ h(x) = f \circ g(x)
\]

for every \( x \) in the domain of \( f \circ g \). What is the domain of \( g \circ h \)?

15. Show that

\[
f(x) = \begin{cases} 
x^2 + 1 & \text{for } x \leq 1 \\
2x & \text{for } x > 1
\end{cases}
\]

is differentiable at 1, but

\[
g(x) = \begin{cases} 
x^2 + 2 & \text{for } x \leq 1 \\
3x & \text{for } x > 1
\end{cases}
\]

is not differentiable at 1.

16. (a) State the precise definition of \( \lim_{x \to \infty} f(x) = L \).

(b) Using the definition, prove that \( \lim_{x \to \infty} \frac{4}{x^2} = 0 \).

17. (a) Suppose \( f(x) = x \cos(x) \) and \( g(x) = xe^x \). Write an expression for \( f \circ g \).

(b) Consider the graph of \( y = f \circ g(x) \) (from part (a)). Suppose that this curve is reflected about the line \( x = 3 \) and shifted four units down. Write an expression for the function whose graph is this new curve.

18. Consider the following argument: I claim that 10 divides \( 11^n + 5 \) for all \( n \). I will prove this by induction. Assume that 10 divides \( 11^k + 5 \). Then I must show that 10 divides \( 11^{k+1} + 5 \). But, \( 11^{k+1} + 5 = 11(11^k + 5) - 50 \). And since 10 divides 50, and by assumption 10 divides \( 11^k + 5 \), this proves the claim. Is my proof correct?

19. (a) Use the definition of derivative to prove that \( \frac{d}{dx}(x^2 + 1) = 2x \).

(b) Using the fact that \( \frac{d}{dx}(e^x) = e^x \), prove that \( \frac{d}{dx}(\ln x) = 1/x \).
20. LIMITS