FINAL EXAM
You have 3 hours.
No notes, no books.
YOU MUST SHOW ALL WORK TO RECEIVE CREDIT
Good luck!

Name

1. __________ (/30 points)

2. __________ (/20 points)

3. __________ (/30 points)

4. __________ (/20 points)

5. __________ (/30 points)

6. __________ (/30 points)

7. __________ (/30 points)

8. __________ (/30 points)

9. __________ (/30 points)

Bonus Question __________ (/20 points)

Total __________ (/250 points)
1. Use the \( \epsilon - \delta \) definition of a limit to prove that

\[
\lim_{x \to 2} (3x - 2) = 4
\]

**Solution:**

We need to show that for any \( \epsilon > 0 \), there exist a \( \delta > 0 \) such that

\[
|x - 2| < \delta \implies |(3x - 2) - 4| < \epsilon
\]

In order to choose this \( \delta \), let’s manipulate the conclusion of the needed implication (the statement on the right) to see if we can find a suggestion.

\[
|(3x - 2) - 4| < \epsilon \iff |3x - 6| < \epsilon \iff |x - 2| < \epsilon/3
\]

Since the hypothesis for our needed implication is \( |x - 2| < \delta \), the above suggests that we should choose \( \delta = \epsilon/3 \).

We now begin our proof:

Let \( \epsilon > 0 \) be given, and choose \( \delta = \epsilon/3 \). Then

\[
|x - 2| < \delta \iff |x - 2| < \epsilon/3 \\
\iff |3x - 6| < \epsilon \\
\iff |(3x - 2) - 4| < \epsilon
\]

Therefore we have that

\[
|x - 2| < \delta \implies |(3x - 2) - 4| < \epsilon
\]

as needed. So, we conclude that

\[
\lim_{x \to 2} (3x - 2) = 4
\]
2. Use the limit definition of derivative to find the derivative of the function $f(x) = x^3 + 5x$.

**Solution:**

By definition, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{((x + h)^3 + 5(x + h)) - ((x)^3 + 5(x))}{h}$$

$$= \lim_{h \to 0} \frac{((x + h)^3 - (x)^3) + 5((x + h) - (x))}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3 - x^3 + 5h}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3 + 5h}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 5$$

$$= 3x^2 + 5$$
3. Find the derivative of \( y = \cot^{-1} x \) by first solving for \( x \), and then taking an implicit derivative of both sides with respect to \( x \). (Recall that \( \cot^{-1} \) is the inverse function for \( \cot \), NOT the reciprocal!) Simplify your answer so that there are no trig functions remaining.

**Solution:**

The solution is analogous to the derivation of the derivative of \( y = \sin^{-1} x \) done in class and in the book.

\[
y = \cot^{-1} x \quad \text{(Solve for \( x \))}
\]

\[
x = \cot y \quad \text{(Take an implicit derivative)}
\]

\[
1 = (- \csc^2 y) \left( \frac{dy}{dx} \right)
\]

\[
\frac{dy}{dx} = \frac{-1}{\csc^2 y} \quad \text{(Use the Pyth. Id.)}
\]

\[
\frac{dy}{dx} = \frac{-1}{1 + \cot^2 y}
\]

\[
\frac{dy}{dx} = \frac{-1}{1 + x^2}
\]
4. The graph below is the graph of \( f'(x) \).

\[
\begin{array}{c}
\text{x = 3} \\
\text{x = -1} \\
\text{x = 1} \\
\text{x = 2} \\
\text{x = 3}
\end{array}
\]

a) What are the intervals of increase and decrease of the function \( f(x) \)?

**Solution:**

The function \( f \) is increasing on the range where \( f' \) is positive; so, we have that the intervals of increase are \((-3, 1)\) and \((3, \infty)\).

Similarly, the intervals of decrease are where \( f' \) is negative; namely, \((-\infty, -3)\) and \((1, 3)\).

b) On what intervals is \( f \) concave up? Concave down?

**Solution:**

The function \( f \) is concave up when \( f'' > 0 \), which is equivalent to saying that \( f' \) must be increasing. So, we have the intervals \((-\infty, -1)\) and \((2, \infty)\).

Similarly, we have that \( f \) is concave down when \( f' \) is decreasing, which occurs on the interval \((-1, 2)\).
5. A spherical balloon is being inflated with air. At the moment when the volume is $36\pi$ in$^3$, the surface area of the balloon in increasing at a rate of $48\pi$ in$^2$/sec.

How fast is the radius of the balloon increasing at that moment?

**Solution:**

![Diagram of a sphere with labels for radius (r), surface area (A), and volume (V).]

We are given a rate of change of the surface area of the balloon; we are interested in the rate of change of the radius of the balloon. So, we first find a relationship between these two variables.

$$A = 4\pi r^2$$

Then, we take the derivative with respect to time, giving us

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{8\pi r} \frac{dA}{dt}$$

Clearly we need to find $r$ at the moment we are interested in; of course, we are given that the volume of the sphere at that moment is $36\pi$ in$^3$. Using $V = \frac{4}{3}\pi r^3$, we conclude that $r$ must at that moment be 3 in.

Plugging all of this into our equation above, we get

$$\frac{dr}{dt} = \frac{1}{8\pi r} \frac{dA}{dt} = \frac{1}{8\pi (3\text{ in})} \left(48\pi \text{ in}^2/\text{sec}\right) = 2\text{ in/sec}$$
6. Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius 1, as shown in the picture below.

![Diagram of a semicircle with a rectangle inscribed](image)

**Solution:**

We label the rectangle as shown in the above diagram – it has height $y$ and width $2x$.

The variable that we want to maximize, $A$, can therefore be written as

$$A = (2x)(y) = 2xy$$

To be able to apply the optimization techniques, we will need to express this as a function of a single variable; so we will have to find a relationship between $x$ and $y$. We notice the right triangle, and thus conclude that

$$x^2 + y^2 = 1 \implies y = \sqrt{1-x^2}$$

We thus rewrite our formula for $A$ as

$$A = 2x\sqrt{1-x^2} \quad x \in [0,1]$$

Taking the derivative, we get

$$A' = (2x) \left( \frac{(1/2)(-2x)}{\sqrt{1-x^2}} \right) + (2)(\sqrt{1-x^2}) = \left( \frac{-2x^2}{\sqrt{1-x^2}} \right) + \frac{(2)(1-x^2)}{\sqrt{1-x^2}} = \frac{2-4x^2}{\sqrt{1-x^2}}$$

Of course we are interested in values of $x$ where this is zero, so we get $2-4x^2 = 0$, which gives us $x = \sqrt{1/2}$. Since this is the only critical point, and since the area is zero at the endpoints (where $x$ is 0 or 1), this must be the value of $x$ that maximizes the area. And of course if $x = \sqrt{1/2}$, then $y = \sqrt{1/2}$. So the dimensions of the rectangle are: width = $2x = 2\sqrt{1/2} = \sqrt{2}$; height = $y = \sqrt{1/2}$. 

7. a) State the Mean Value Theorem (remember to include all the necessary details!)

Solution:

The Mean Value Theorem: Let \( f \) be a differentiable function on the interval \([a, b]\). Then there exists a point \( c \in (a, b) \) with the property that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

b) Use the Mean Value Theorem to show that if a differentiable function \( f \) has roots at \( x = a \) and \( x = b \), then \( f \) must have a critical point somewhere in \((a, b)\).

Solution:

The given function \( f \) is differentiable, so we can apply the Mean Value Theorem. This tells us that there exists a point \( c \in [a, b] \) with

\[
f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0
\]

Since \( f'(c) = 0 \), we have that \( c \) is a critical point, and thus the claim is true.
8. a) Calculate the following integral using a Riemann sum with right endpoints:

\[ \int_0^5 (x - 2) \, dx \]

Solution:

We have \( a = 0, \) and \( b = 5, \) so \( \Delta x = 5/n, \) and \( x_i = a + i\Delta x = 5i/n. \) So

\[
\int_0^5 (x - 2) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} (x_i - 2) \Delta x \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{5i}{n} - 2 \right) \frac{5}{n} \]

\[ = \lim_{n \to \infty} \frac{5}{n} \left[ \sum_{i=1}^{n} \left( \frac{5i}{n} - 2 \right) \right] \]

\[ = \lim_{n \to \infty} \frac{5}{n} \left[ \frac{5n}{n} \left( \sum_{i=1}^{n} i \right) - \left( \sum_{i=1}^{n} 2 \right) \right] \]

\[ = \lim_{n \to \infty} \frac{25}{n^2} \cdot \frac{n(n+1)}{2} - \frac{5}{n} \cdot 2n \]

\[ = \lim_{n \to \infty} \frac{25n^2 + 25}{2n^2} - 10 \]

\[ = \lim_{n \to \infty} \frac{25 + 25n^{-2}}{2} - 10 \]

\[ = \frac{5}{2} \]

b) Calculate the same integral by interpreting it as “signed area” between the graph of the function and the \( x \)-axis.

\[
\text{Integral} = A_2 - A_1
\]

\[ = \frac{1}{2}(3)(3) - \frac{1}{2}(2)(2) \]

\[ = 5/2 \]
9. a) Find the derivatives of \((\sin x)\) and \((x \cos x)\).

**Solution:**

\[
\frac{d}{dx}(\sin x) = \cos x \\
\frac{d}{dx}(x \cos x) = (x)'(\cos x) + (x)(\cos x)' = \cos x - x \sin x
\]

b) Use the above formulas to find an antiderivative \(F(x)\) for \(f(x) = x \sin x\).

**Solution:**

Note above that \(x \sin x\) appears in the second expression; and the only other term in that expression is exactly the derivative from the first expression... So, we notice that

\[
\left(\sin x - x \cos x\right)' = (\cos x) - (\cos x - x \sin x) = x \sin x
\]

So, we conclude that an antiderivative for \(f(x) = x \sin x\) is \(F(x) = \sin x - x \cos x\).

c) Use the answer to part (b) to determine the area between the graph of \(y = x \sin x\) and the \(x\)-axis between \(x = 0\) and \(x = \pi\).

**Solution:**

We note that \(f(x) = x \sin x\) is positive for all values of \(x\) between 0 and \(\pi\); so, the area in question is merely an integral. Using the Evaluation Theorem, we have

\[
\int_0^\pi x \sin x \, dx = \left[ f(x) \right]_0^\pi = F(\pi) - F(0) = (\sin \pi - \pi \cos \pi) - (\sin 0 - 0 \cos 0) \\
= (0 - \pi(-1)) - (0 - 0) \\
= \pi
\]
**Bonus Question:**

Show that for any function $f(x)$ defined on an interval $[a, b]$, 

$$
\lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n
$$

whenever these limits exist. (Recall that $L_n$ and $R_n$ are, respectively, the left-endpoint and right-endpoint approximations to the area under the graph of $f$.)

**Solution:**

By definition, we have

$$
L_n = \sum_{i=0}^{n-1} f(x_i) \Delta x
$$

$$
R_n = \sum_{i=1}^{n} f(x_i) \Delta x
$$

So, we have that

$$
\lim_{n \to \infty} L_n - \lim_{n \to \infty} R_n = \lim_{n \to \infty} (L_n - R_n)
$$

$$
= \lim_{n \to \infty} \left( \sum_{i=0}^{n-1} f(x_i) \Delta x - \sum_{i=1}^{n} f(x_i) \Delta x \right)
$$

$$
= \lim_{n \to \infty} \left( f(x_0) \Delta x - f(x_n) \Delta x \right)
$$

$$
= \lim_{n \to \infty} \left( f(x_0) - f(x_n) \right) \Delta x
$$

$$
= \lim_{n \to \infty} \left( f(a) - f(b) \right) \frac{b-a}{n}
$$

$$
= \lim_{n \to \infty} \frac{f(a) - f(b)}{n} \left( b - a \right)
$$

$$
= \left( f(a) - f(b) \right) \frac{b-a}{n} \lim_{n \to \infty} \frac{1}{n}
$$

$$
= 0
$$

Since the difference of the limits is zero, we conclude that the limits are equal.