EXAM 3

Math 216, 2021 Spring.

Name: Solutions NetID:________ Student ID:________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. *(16 pts)* Of the matrices below, there is exactly one pair of them that are similar to each other.

$$M_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(a) Identify the similar pair, and explain how you know the others are not similar.

All of these are Jordan forms, which are similar iff they are block rearrangements.

So M_1, M_4 are the only similar pair.

(b) Name one of these matrices A and the other B, and find a matrix C for which $A = CBC^{-1}$.

$$A = M_1 = [T]_{\mathcal{V}}^{\mathcal{V}}, \quad \mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$$

$$B = M_4 = [T]_{\mathcal{W}}^{\mathcal{W}}, \quad \mathcal{W} = \{\underset{\overset{\uparrow}{\vec{v}_4}}{\vec{w}_1}, \underset{\overset{\uparrow}{\vec{v}_1}}{\vec{w}_2}, \underset{\overset{\uparrow}{\vec{v}_2}}{\vec{w}_3}, \underset{\overset{\uparrow}{\vec{v}_3}}{\vec{w}_4}\}$$

$$A = CBC^{-1}$$

$$M_1 = CM_4C^{-1}$$

$$[T]_{\mathcal{V}}^{\mathcal{V}} = [I]_{\mathcal{W}}^{\mathcal{V}} [T]_{\mathcal{W}}^{\mathcal{W}} [I]_{\mathcal{V}}^{\mathcal{W}}$$

$$C = [I]_{\mathcal{W}}^{\mathcal{V}} = \begin{pmatrix} | & | & | & | \\ [\vec{w}_1]_{\mathcal{V}} & [\vec{w}_2]_{\mathcal{V}} & [\vec{w}_3]_{\mathcal{V}} & [\vec{w}_4]_{\mathcal{V}} \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

2. *(17 pts)* The vector space $\mathbb{R}^3$ is made an inner product space V using the non-standard inner product $\langle \vec{v}, \vec{w} \rangle = [\vec{v}]_{\mathcal{V}} \cdot [\vec{w}]_{\mathcal{V}}$, where

$$\mathcal{V} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\vec{v}$, $\vec{w}$

(a) Find the angle in V between $(0, 1, 0)$ and $(0, 0, 1)$. *(Hint: Some of the arithmetic given in the statement of question 4. on this exam might be useful.)*

$$[\vec{x}]_{\mathcal{V}} = [I]^{\mathcal{V}}_{\mathcal{S}} [\vec{x}]_{\mathcal{S}} = \left([I]^{\mathcal{S}}_{\mathcal{V}}\right)^{-1} [\vec{x}]_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}^{-1} \vec{x} = \begin{pmatrix} -3 & 9 & -5 \\ 1 & -3 & 2 \\ 1 & -2 & 1 \end{pmatrix} \vec{x}$$

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(9,-3,-2)\cdot(9,-3,-2)} = \sqrt{94}$$

$$\|\vec{w}\| = \sqrt{\langle \vec{w}, \vec{w} \rangle} = \sqrt{(-5,2,1)\cdot(-5,2,1)} = \sqrt{30}$$

$$\langle \vec{v}, \vec{w} \rangle = (9,-3,-2)\cdot(-5,2,1) = -53$$

$$\theta = \arccos \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \, \|\vec{w}\|} = \arccos \frac{-53}{\sqrt{94}\sqrt{30}}$$

(b) Find a vector (in coordinates with respect to the standard basis) orthogonal in V to $(4, 1, 2)$.

$$\langle (4,1,2), \vec{x} \rangle = [(4,1,2)]_{\mathcal{V}} \cdot [\vec{x}]_{\mathcal{V}}$$
$$= (-13, 5, 4) \cdot [\vec{x}]_{\mathcal{V}} = 0$$

Choose $[\vec{x}]_{\mathcal{V}} = (0, 4, -5)$. Then

$$[\vec{x}]_{\mathcal{S}} = [I]^{\mathcal{S}}_{\mathcal{V}} \begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} -11 \\ 3 \\ 7 \end{pmatrix}$$

3. *(17 pts)* The function f is a linear combination of $\sin x$ and $\cos x$, and we also know:

$$\int_{-\pi}^{\pi} (x^2+1) f(x) \cos x \, dx = 2 \qquad \int_{-\pi}^{\pi} (x^2+1) \cos^2 x \, dx = \frac{3\pi}{2} + \frac{\pi^3}{3} = k_1$$

$$\int_{-\pi}^{\pi} (x^2+1) f(x) \sin x \, dx = 3 \qquad \int_{-\pi}^{\pi} (x^2+1) \sin^2 x \, dx = \frac{\pi}{2} + \frac{\pi^3}{3} = k_2$$

Identify and use a relevant inner product and an orthonormal basis of span$(\sin x, \cos x)$ to find the function f (you may leave the coefficients in terms of k_1 and k_2).

Choose $\langle p, q\rangle = \int_{-\pi}^{\pi} (x^2+1) p(x) q(x) \, dx$, on span$(\cos x, \sin x)$.

The givens become

$$\langle f, \cos x\rangle = 2 \qquad \|\cos x\|^2 = k_1$$

$$\langle f, \sin x\rangle = 3 \qquad \|\sin x\|^2 = k_2$$

Then $v(x) = \frac{1}{\sqrt{k_1}} \cos x$, $w(x) = \frac{1}{\sqrt{k_2}} \sin x$ are unit vectors, and

$$\langle v(x), w(x)\rangle = \int_{-\pi}^{\pi} (x^2+1)\left(\frac{1}{\sqrt{k_1}} \cos x\right)\left(\frac{1}{\sqrt{k_2}} \sin x\right) dx = 0$$

by symmetry because the integrand is odd and the domain is symmetric. So $\{v, w\}$ is an orthonormal basis.

$$\langle f, v\rangle = \langle f, \tfrac{1}{\sqrt{k_1}} \cos x\rangle = \tfrac{1}{\sqrt{k_1}} \langle f, \cos x\rangle = \frac{2}{\sqrt{k_1}}$$

$$\langle f, w\rangle = \langle f, \tfrac{1}{\sqrt{k_2}} \sin x\rangle = \tfrac{1}{\sqrt{k_2}} \langle f, \sin x\rangle = \frac{3}{\sqrt{k_2}}$$

$$\Rightarrow f = \left(\frac{2}{\sqrt{k_1}}\right) v + \left(\frac{3}{\sqrt{k_2}}\right) w$$

$$= \left(\frac{2}{\sqrt{k_1}}\right)\left(\frac{1}{\sqrt{k_1}} \cos x\right) + \left(\frac{3}{\sqrt{k_2}}\right)\left(\frac{1}{\sqrt{k_2}} \sin x\right)$$

$$= \left(\frac{2}{k_1}\right) \cos x + \left(\frac{3}{k_2}\right) \sin x$$

4. *(17 pts)* The information below is given. Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$, and the solution to the initial value problem with $\vec{y}(0) = (1, 0, 0)$.

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} -3 & 1 & 1 \\ 9 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 & 1 \\ 9 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix}$$

This shows that these are inverses, so this is a conjugation. So this is the Jordan form for A.

Using a Jordan basis $\mathcal{V}$, the conjugation becomes

$$[T]_{\mathcal{V}}^{\mathcal{V}} = [I]_{\mathcal{S}}^{\mathcal{V}} [T]_{\mathcal{S}}^{\mathcal{S}} [I]_{\mathcal{V}}^{\mathcal{S}}$$

So $P = [I]_{\mathcal{V}}^{\mathcal{S}} = \begin{pmatrix} -3 & 1 & 1 \\ 9 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix}$ has columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$

that make such a Jordan basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Then a fundamental set of solutions is

$$e^{xA}\vec{v}_1 = e^{3x}\vec{v}_1 = e^{3x}\begin{pmatrix} -3 \\ 9 \\ -5 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{3x}(\vec{v}_2 + x\vec{v}_1) = e^{3x}\begin{pmatrix} 1-3x \\ -3+9x \\ 2-5x \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{5x}\vec{v}_3 = e^{5x}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The solution to the I.V.P. is

$$\vec{y} = e^{xA}\vec{e}_1$$

We know $e^{xA}\vec{v}_i$, so we need to find c_1, c_2, c_3 with $\vec{e}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

$$\vec{e}_1 = \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}}_{P}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Longrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix}}_{P^{-1}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Then

$$e^{xA}\vec{e}_1 = e^{xA}\left(1\vec{v}_1 + 1\vec{v}_2 + 3\vec{v}_3\right)$$

$$= e^{xA}\vec{v}_1 + e^{xA}\vec{v}_2 + 3e^{xA}\vec{v}_3$$

$$= e^{3x}\begin{pmatrix} -3 \\ 9 \\ -5 \end{pmatrix} + e^{3x}\begin{pmatrix} 1-3x \\ -3+9x \\ 2-5x \end{pmatrix} + 3e^{5x}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= e^{3x}\begin{pmatrix} -2-3x \\ 6+9x \\ -3-5x \end{pmatrix} + e^{5x}\begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$$

5. *(17 pts)* Find the form of a particular solution to the equation below. Don't evaluate the coefficients, but explain how you know they can be found.

$$\vec{y}' = A\vec{y} + \begin{pmatrix} e^x \\ x \\ 1 \end{pmatrix}, \text{ with } A = \begin{pmatrix} 4 & 2 & 12 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

We rewrite as $\vec{y}' = A\vec{y} + e^x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and guess $\vec{y}_p = e^x\vec{a} + x\vec{b} + \vec{c}$. Plugging in we get

$$e^x\vec{a} + \vec{b} = A\left(e^x\vec{a} + x\vec{b} + \vec{c}\right) + e^x\vec{e}_1 + x\vec{e}_2 + 1\vec{e}_3$$

$$e^x\underbrace{\left(A\vec{a} + \vec{e}_1 - \vec{a}\right)}_{=\vec{0}} + x\underbrace{\left(A\vec{b} + \vec{e}_2\right)}_{=\vec{0}} + 1\underbrace{\left(A\vec{c} + \vec{e}_3 - \vec{b}\right)}_{=\vec{0}} = \vec{0}$$

$$\Rightarrow \begin{cases} (A - 1I)\vec{a} = -\vec{e}_1 & (1) \\ A\vec{b} = -\vec{e}_2 & (2) \\ A\vec{c} = \vec{b} - \vec{e}_3 & (3) \end{cases}$$

We can solve (1) for $\vec{a}$ because $\det(A - 1I) = 6 \neq 0$.

We can solve (2) for $\vec{b}$ because $\det A = 24 \neq 0$.

We can then use $\vec{b}$ to solve (3) for $\vec{c}$ because $\det A = 24 \neq 0$.

6. *(16 pts)* Your friend Bob says that he has found an example of a 3rd order constant coefficient linear homogeneous differential equation whose characteristic polynomial has a real root with multiplicity 2, and that when converted to a first order system of equations the resulting coefficient matrix is diagonalizable.

Find such an example (showing that Bob could be right) or explain how you know Bob must be wrong.

Because of the repeated root, the original CCLDE must have a solution of the form

$$y = xe^{rx}$$

If the converted first order system is diagonalizable then its general solution will be of the form

$$\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \vec{u} = c_1 e^{\lambda_1 x}\begin{pmatrix} a_1 \\ b_1 \\ d_1 \end{pmatrix} + c_2 e^{\lambda_2 x}\begin{pmatrix} a_2 \\ b_2 \\ d_2 \end{pmatrix} + c_3 e^{\lambda_3 x}\begin{pmatrix} a_3 \\ b_3 \\ d_3 \end{pmatrix}$$

This would mean then that the general solution to the original CCLDE would be

$$y = c_1 a_1 e^{\lambda_1 x} + c_2 a_2 e^{\lambda_2 x} + c_3 a_3 e^{\lambda_3 x}$$

But this does not include the known solution previously observed.

So Bob must be wrong.