EXAM 2

Math 216, 2021 Spring.

Name: Solutions NetID:________ Student ID:________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. *(16 pts)* Find a fundamental set of real solutions to the constant coefficient linear differential equation $L(y) = 0$ whose characteristic polynomial is

$$p(\lambda) = (\lambda + 1)(\lambda - 2)^3\underbrace{(\lambda^3 + 5\lambda^2 + 17\lambda + 13)}_{q(x)}{}^2$$

$q(x)$ has possible rational roots $\pm 1, \pm 13$.

$q(-1) = 0 \Rightarrow (\lambda+1)$ is a factor.

$$\begin{array}{r|l}
 & \lambda^2 + 4\lambda + 13 \\
\hline
\lambda + 1 & \lambda^3 + 5\lambda^2 + 17\lambda + 13 \\
 & \underline{\lambda^3 + 1\lambda^2} \\
 & \quad 4\lambda^2 + 17\lambda + 13 \\
 & \quad \underline{4\lambda^2 + 4\lambda} \\
 & \qquad\quad 13\lambda + 13 \\
 & \qquad\quad \underline{13\lambda + 13} \\
 & \qquad\qquad\quad 0
\end{array}$$

$q_2(\lambda) = (\lambda+2)^2 + 9$

has roots $-2 \pm 3i$

So

$$p(\lambda) = (\lambda+1)^3(\lambda-2)^3\big(\lambda-(-2+3i)\big)^2\big(\lambda-(-2-3i)\big)^2$$

and a fundamental set of real solutions is

$$\left\{ e^{-x}, xe^{-x}, x^2e^{-x}, e^{2x}, xe^{2x}, x^2e^{2x}, \right.$$

$$\left. e^{-2x}\cos(3x), e^{-2x}\sin(3x), xe^{-2x}\cos(3x), xe^{-2x}\sin(3x) \right\}$$

2. *(17 pts)* Find the form of a particular solution to the constant coefficient linear differential equation

$$L(y) = \underbrace{6xe^x \sin(3x)}_{g_1} - \underbrace{x^2 e^{4x}}_{g_2}$$

whose characteristic polynomial is

$$p(\lambda) = (\lambda - 4)^7(\lambda^2 + 5)\underbrace{(\lambda^2 - 2\lambda + 10)}_{\text{has roots } 1 \pm 3i}(4\lambda + 2)$$

For g_1, $r = a+bi = 1+3i$, is a root of p with $m=1$.

So we have $y_{p_1} = x(a_1x+a_0)e^x\cos(3x)$
$+ x(b_1x+b_0)e^x\sin(3x)$.

For g_2, $r = a+bi = 4$, is a root of p with $m=7$.

So we have $y_{p_2} = x^7(c_2x^2+c_1x+c_0)e^{4x}$.

Then $y_p = y_{p_1} + y_{p_2}$

$$\begin{aligned} &= x(a_1x+a_0)e^x\cos(3x) \\ &\quad + x(b_1x+b_0)e^x\sin(3x) \\ &\quad + x^7(c_2x^2+c_1x+c_0)e^{4x} \end{aligned}$$

3. *(16 pts)* In this question we have a 1 kg mass, and a hanging spring which stretches 2 m when the mass is attached. Gravitational acceleration in this question is 10 m/s^2 and there is no friction.

(a) Find a fundamental set of real solutions to the differential equation describing the above system.

$$F = mg = kx \quad \Rightarrow (1\,\mathrm{kg})\left(10\,\tfrac{m}{s^2}\right) = (k)(2m)$$
$$\Rightarrow \quad k = 5\left(\mathrm{kg}/s^2\right)$$

$1y'' + 5y = 0$ has $p(\lambda) = \lambda^2 + 5$, roots $r = \pm i\sqrt{5}$.

So a fundamental set is $\{\cos(\sqrt{5}x), \sin(\sqrt{5}x)\}$.

(b) Suppose now that an external periodic force of $\cos(4t)$ N is applied. Find a particular solution and identify if there is resonance. (Recall 1 N = 1 kg m/s^2.)

$p(\lambda) = \lambda^2 + 5 \leftarrow 1y'' + 5y = \cos(4t) \rightarrow r = 0 + 4i$ is not a root of p.

We guess $y_p = A\cos(4t) + B\sin(4t)$

$$y_p'' = -16A\cos(4t) - 16B\sin(4t)$$

The equation becomes

$$\left(-16A\cos(4t) - 16B\sin(4t)\right) + 5\left(A\cos(4t) + B\sin(4t)\right) = \cos(4t)$$

$$-11A\cos(4t) - 11B\sin(4t) = 1\cos(4t) + 0\sin(4t)$$

$$\Rightarrow A = \frac{-1}{11},\ B = 0$$

So $y_p = \frac{-1}{11}\cos(4t)$, which is <u>not</u> resonant.

4. *(17 pts)* Let V be the vector space of continuously differentiable vector fields in $\mathbb{R}^3$, let B be the solid closed unit ball in $\mathbb{R}^3$, let g be a continuous real-valued function on $\mathbb{R}^3$, and define $P_g : V \to \mathbb{R}$ by

$$P_g(\vec{F}) = \iiint_B \left(\nabla \cdot \vec{F}(\vec{x})\right)\left(g(\vec{x})\right) dV$$

Bob says that P_g is a linear transformation; prove he is right, or find a counterexample to show he is wrong.

$$P_g\left(a\vec{F} + b\vec{G}\right) = \iiint_B \nabla \cdot \left(a\vec{F} + b\vec{G}\right) g(x)\, dV$$

$$= \iiint_B \left(a\nabla \cdot \vec{F} + b\nabla \cdot \vec{G}\right) g(x)\, dV$$

$$= \iiint_B \left(a\nabla \cdot \vec{F}\, g(x) + b\nabla \cdot \vec{G}\, g(x)\right) dV$$

$$= a\iiint_B \nabla \cdot \vec{F}\, g(x)\, dV + b\iiint_B \nabla \cdot \vec{G}\, g(x)\, dV$$

$$= a P_g(\vec{F}) + b P_g(\vec{G}).$$

So P_g is a linear transformation.

5. *(17 pts)* There are values of b and c such that for every y that is a solution to the differential equation

$$y^{[5]} + 6y^{[4]} + 11y''' + 10y'' + 13y' + 3y = 0$$

then $by' + cy$ is a solution to the differential equation

$$y^{[4]} + 3y''' + 2y'' + 4y' + y = 0.$$

Use linear differential operators to find these values, WITHOUT plugging in directly OR solving any system of equations.

We rewrite the given as

$$(D^5 + 6D^4 + 11D^3 + 10D^2 + 13D + 3)y = 0$$

$$\Downarrow$$

$$(D^4 + 3D^3 + 2D^2 + 4D + 1)(bD + c)y = 0$$

This works for example if $(bD+c)$ is the quotient. since in that case the equations are the same.

$$\begin{array}{r|l} & 1D + 3 \\ \hline D^4 + 3D^3 + 2D^2 + 4D + 1 & D^5 + 6D^4 + 11D^3 + 10D^2 + 13D + 3 \\ & \underline{D^5 + 3D^4 + 2D^3 + 4D^2 + 1D} \\ & 3D^4 + 9D^3 + 6D^2 + 12D + 3 \\ & \underline{3D^4 + 9D^3 + 6D^2 + 12D + 3} \\ & 0 \end{array}$$

So we can choose $b=1$, $c=3$.

$$\overset{f_1}{\shortparallel}\ \overset{f_2}{\shortparallel}\ \overset{f_3}{\shortparallel} \qquad \overset{g_1}{\shortparallel}\ \overset{g_2}{\shortparallel}\ \overset{g_3}{\shortparallel}$$

6. *(17 pts)* In P_2 we consider the bases $\mathcal{S} = \{1, x, x^2\}$ and $\mathcal{V} = \{1, (x-2), (x-2)^2\}$. Let $T : P_2 \to P_2$ be the linear transformation defined by $T(p)(x) = \frac{d}{dx}\Big((x-2)p(x)\Big)$ (so, given a polynomial p, multiply by $(x-2)$ and then take the derivative to get $T(p)$).

(a) Find the matrix $M = [T]^{\mathcal{V}}_{\mathcal{V}}$.

$$T(g_1) = ((x-2)\cdot 1)' = 1 \qquad = 1g_1$$

$$T(g_2) = ((x-2)(x-2))' = 2(x-2) \qquad = 2g_2$$

$$T(g_3) = ((x-2)(x-2)^2)' = 3(x-2)^2 \qquad = 3g_3$$

$$[T]^{\mathcal{V}}_{\mathcal{V}} = \begin{pmatrix} | & | & | \\ [T(g_1)]_{\mathcal{V}} & [T(g_2)]_{\mathcal{V}} & [T(g_3)]_{\mathcal{V}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) Find the change of basis matrices $[I]^{\mathcal{S}}_{\mathcal{V}}$ and $[I]^{\mathcal{S}}_{\mathcal{V}}$.

$$[I]^{\mathcal{S}}_{\mathcal{V}} = \begin{pmatrix} | & | & | \\ [g_1]_{\mathcal{S}} & [g_2]_{\mathcal{S}} & [g_3]_{\mathcal{S}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} 1 + 2\,2 \\ 2 \\ 3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} 1 + 4\,3 \\ 2 + 4\,3 \\ 3 \end{array}$$

$$[I]^{\mathcal{V}}_{\mathcal{S}} = \left([I]^{\mathcal{S}}_{\mathcal{V}}\right)^{-1} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Use the results from the previous two parts to find the matrix $A = [T]_{\mathcal{S}}^{\mathcal{S}}$.

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{B}}^{\mathcal{S}} [T]_{\mathcal{B}}^{\mathcal{B}} [I]_{\mathcal{S}}^{\mathcal{B}}$$

$$= \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{pmatrix}$$